**Teachers' meanings for average rate of change in U.S.A. and Korea**

Hyunkyoung Yoon, Arizona State University
Cameron Byerley, Arizona State University
Patrick W. Thompson, Arizona State University

*This study explores teachers' meanings for average rate of change in U.S.A. and Korea. We believe that teachers convey their meanings to students and teachers who have productive mathematical meanings help students build coherent meanings. We administered a diagnostic instrument to 96 U.S. teachers and 66 Korean teachers. Some of teachers' responses revealed particular problematic meanings for average rate of change that should be addressed in professional development. Our analyses suggest that Korean teachers' meanings for average rate of change are substantially stronger than U.S. teachers' meanings.*

*Key words:* Average rate of change, Mathematical meanings for teaching, Secondary teachers, International comparisons

There has been substantial interest in comparing student and teacher performance in the United States to other countries (Cai, 1995; Ma, 1999; Tatto, Ingvarson et al., 2008). Many people are aware that U.S. students are outperformed on mathematics assessments by students in many Asian countries. It is more surprising that according to PISA (Program for International Student Assessment) and NAEP (National Assessment of Educational Progress) that white students in our best performing state, Massachusetts, did not do as well as the average student in Korea (Hanushek, Peterson, & Woessmann, 2010). Furthermore, the average Korean student from any background outperformed students in Massachusetts who had at least one college educated parent. It is not easy to explain Korean students' superior performance by pointing to substantial diversity in the United States

Studies have demonstrated that there is a positive relationship between teacher knowledge and student performance (Baumert, Kunter et al., 2010; Hill, Ball et al., 2007). The TEDS-M (Teacher Education and Development Study in Mathematics) study investigated differences in teachers' mathematical content knowledge in seventeen countries to give further information about the relationship between teachers' knowledge and student performance internationally (Tatto, Peck et al., 2012). Although Korean teachers were not included in the TEDS-M study, secondary teachers in the United States did have lower scores than secondary teachers in other high performing Asian countries such as Singapore. TIMSS (Trend in International Mathematics and Science Study) and PISA scores indicated that Korean students outperformed other countries in international assessments. However, there are few studies that reveal Korean teachers' knowledge (Kim, 2007).

Our research team developed the *Mathematical Meanings for Teaching Secondary Mathematics* (MMTsm), a 44 item diagnostic instrument designed primarily to give professional developers insight into mathematical meanings with which teachers operate. We have piloted the MMTsm with 460 high school mathematics teachers in the United States. The MMTsm contains items that assess teachers' meanings for variation and covariation, function, proportionality, rate of change, and structure sense (Byerley & Thompson, 2014; Thompson, 2015; Yoon, Hatfield, & Thompson, 2014). In the summer of 2014 the first author translated the instrument into Korean and administered 42 items[1] to a convenience sample of 66 Korean teachers who taught 7th to 12th grades. The goal of the pilot in Korea was to understand how well the items revealed Korean teachers' meanings, to unearth any issues in the item translations, and to generate hypothesis about similarities and differences between Korean and U.S. teachers. As such, we have three research questions:

1) Do the translated versions of items make sense to Korean teachers in the way we intended?

2) What are the Korean teachers' mathematical meanings in the areas that the MMTsm asseses?

3) What are similarities and differences in U.S. and Korean teachers' meanings for average rate of change?

Our study of 7th to 12th grade teachers in Korea and the United States contributes to the investigation of international differences in teacher knowledge in two ways:

1) The MMTsm provides insight into the productive and unproductive meanings teachers operate with instead of categorizing responses as right or wrong.

2) Beyond a few studies with small sample sizes, little is published about secondary teachers' meanings of average rate of change in either the U.S. or Korea.

## Literature Review

The third author (Thompson, 1994b) conducted a teaching experiment on the Fundamental Theorem of Calculus with 19 senior and graduate mathematics students, many of whom planned to teach secondary mathematics. He attended to the concept of average rate of change explicitly in his teaching experiment because of its centrality in understanding difference quotients and the Fundamental Theorem of Calculus. Thompson (1994b) described a typical mature meaning for average rate of change:

> *[By "average rate of change"] we typically mean that if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred. An average speed of 55 km/hr on a trip means that if we were to repeat the trip traveling at a constant rate of 55 km/hr, then we would travel precisely the same amount of distance in precisely the same amount of time as had been the case originally (p. 50).*

Based on quizzes and transcribed recordings of class discussions and tutoring sessions he concluded, the university mathematics students "apparently did not have operational schemes for average rate of change" (p. 49).

Coe (2007) conducted an interview-based study of three secondary teachers' meanings for rate of change. Peggy, an experienced teacher with an undergraduate degree in mathematics, was unable to provide a definition for average rate of change and became confused about how to account for varying speeds in the middle of a trip. The study found, "not one of the [three] teachers evidenced a fully coherent model of thinking that allowed them to work with the average rate tasks" (Coe, 2007, p. 237). If one thinks of speed as a multiplicative comparison of the changes in distance and time, it is possible to imagine an average speed as the constant speed that one must travel to go the same distance in the same amount of time. However, Peggy was

[1] We removed two items that did not work because of differences in language.

inclined to think that speed is an index of "fastness", so all of the changes in speed throughout the trip might seem important to take into consideration.

Additional U.S. studies of calculus students and secondary teachers are related to teachers' understandings of rate of change (Bowers & Doerr, 2001; Stump, 1999; Weber & Dorko, 2014). These studies suggest that teachers' meanings for rate of change might be inadequate for making sense of average rate of change. For example, Bowers and Doerr (2001) investigated 26 secondary teachers' thinking about the "mathematics of change" in two university technology based mathematics classes. They designed the first two instructional sequences to help the participants understand the Fundamental Theorem of Calculus by exploring relationships between linked velocity and position graphs (Bowers & Doerr, 2001, p. 120). Given a non-constant velocity versus time graph, more than seven teachers found the total distance traveled by simply multiplying time elapsed by the velocity at the end of the time interval using the formula $d=rt$. The formula $d=rt$ only works in situations with constant rates of change because the formula reflects a proportional relationship between distance traveled and time elapsed. Technically, this formula should be written "$\Delta d=r\Delta t$", because "$d=rt$" is only true if distance and time are both measured from zero. This misapplication of $d=rt$ suggests that teachers do not have an image of constant speed as a proportional relationship between changes in distance traveled and changes in elapsed time—an understanding of constant speed that is productive in developing a mature meaning of average rate of change.

Weber & Dorko (2014) investigated calculus students' and professors' descriptions of rate of change in various calculus situations. The meanings students displayed did not depend on making multiplicative comparisons of the change in one quantity to the associated change in another. For example, students conveyed meanings such as "rate as the process of differentiating a function, defined algebraically, using rules (e.g. product rule)", "rate as the slant or steepness of a graph" and "rate as something a function (or object) possesses (e.g. weight)" (p. 23). These meanings for rate do not involve relative size of changes and do not support a mature meaning of average rate of change. It is by thinking about the proportional relationship between changes in distance and changes in time that one sees why an understanding of "average rate of change" as arithmetic mean does not work. The mathematics professors in Weber & Dorko's study were much more likely to describe rate as "measuring the simultaneous variation of variable, or how fast variables change with respect to each other" (p. 23).

The first author did not find any studies about Korean teachers' meanings for rate of change or average rate of change after searching in Korean and English. However, Cho (2010) found that 36 Korean high school mathematics teachers showed high Mathematical Knowledge for Teaching (MKT) for Differentiation. The Korean teachers demonstrated particularly high subject matter knowledge on Cho's instrument that included a task on average rate of change. TEDS-M did not release any information on secondary teachers' understandings of rate of change (Tatto & Senk, 2011).

## Theoretical Perspective

Coherent mathematical meanings serve as a foundation for future learning, so it is important that students build useful and robust meanings. One way students develop meanings is by trying to make sense of what their teacher say and do in the classroom. Before discussing how meanings are conveyed in the classroom, we will explain what we mean by *meanings*. According to Piaget, to understand is to assimilate to a scheme (Skemp, 1962, 1971; Thompson, 2013;

Thompson & Saldanha, 2003). Thus, the phrase "a person attached a meaning to a word, symbol, expression, or statement" means that the person assimilated the word, symbol, expression, or statement to a scheme. A scheme is an organization of ways of thinking, images, and schemes. When we say *assimilate* we mean the ways in which an individual interprets and make sense of a text, utterance, or self-generated thought. According to Piaget, repeated assimilation is the source of schemes, and new schemes emerge through repeated assimilations, which early on require functional accommodations and eventually entail metamorphic accommodations (Steffe, 1991).

We focus on teachers' mathematical meanings because of their centrality in students' construction of meaning. In classrooms, students might construct their meanings from their peers, from prior schemes, from resources the teacher selects for them or resources they find on their own. However, we suspect that a main source of students' mathematical meanings lies in what teachers say and do. Students try to assimilate what the teacher says and does using their understandings of what is being taught. In doing so, the students will adjust what they say and do according to their understanding of what their teacher intends. In this sense, conversations in the classroom between a teacher and students entail mutual attempts by the teacher and students to understand each other. We suspect that teachers exert less effort in this regard than do students, and hence teachers have a greater impact on students' meanings than do students have on the teacher's meanings.

Our theory of meaning, and of ways meanings are conveyed through mutual interpretation, allows us to bridge theoretically what teachers know, what they teach, and what their students learn. While we cannot access the teachers' mathematical meanings directly, we can delimit categories of responses according to particular mathematical meanings that we discern from them. We categorize teachers' response based on meanings we believe might underlie the response based on the best available evidence of interviews and prior qualitative work. We assumed that, for the most part, meanings that teachers used to construct their responses to an item are meanings that would guide their decisions in the classroom.

We believe that meanings students construct are related to but not identical to a teachers' meanings. In other words, there might be some gap between what teachers have in mind and what students understand. For example, a teacher might define average speed for her students by writing down a formula, but understand that the formula is related to finding a constant speed a hypothetical object would need to travel to go the same distance in the same amount of time. However, her students might understand that average rate of change is a formula to be applied in situations with the key words "average rate of change", failing to develop a quantitatively rich meaning for average rate of change that helps them use it in a variety of contexts. Knowing a normatively correct mathematical formula for average rate of change is not the same as having a productive meaning for the formula. We found many teachers who were able to write a normatively correct formula to compute average rate of change on one item (not discussed in this paper), but who were unable to use it productively to answer the two items discussed later in this paper. Our focus on teachers' meanings as a root for their actions allows us to think of meanings we think students might construct based on meanings we attribute to teachers. For example, if the teacher conveys the meaning that average rate of change is a formula, we believe the students might only construct a meaning for average rate of change as a formula that should be used in particular situations.

## Methodology

Thompson (2015, p. 979) explained the process of creating items and rubrics for the *MMTsm*. We summarize the steps of a three year process below:

1) Draft items, interview teachers, and give item to mathematicians and math educators for review.
2) Revise items, interview teachers again.
3) Administer items to large sample of teachers and analyze responses in terms of the meanings they revealed.
4) Retire unusable items.
5) Interview teachers to understand why they gave the response that they did.
6) Revise items, potentially using teacher responses to make items multiple choice.
7) Administer revised items to large sample
8) Develop scoring rubrics.

After a first round of data collection in 2012, we categorized the responses from 144 teachers using a modified grounded theory approach (Corbin & Strauss, 2007). The modification was that we began our data analysis with strong theories of understanding magnitudes and rates of change, and of the nature of mathematical meanings and of characteristics that make them productive in instruction. After the 2013 pilot with revised items we developed a scoring rubric for each item by grouping grounded codes into levels based on the quality of the mathematical meanings expressed. The 96 U.S. high school teachers' responses reported in this paper are from the 2013 pilot. During team discussions of rubrics and responses, we continually asked ourselves "how productive would meanings we can discern from the teacher's response be for a student were the teacher to convey it?"

The first author translated each item into Korean. A Korean mathematics Ph.D. student, who taught high school mathematics in Korea for 7 years and wrote items for the Korean version of the practice SAT, translated the items back into English. The Ph.D. student had never seen the English versions. The first author and the third author reviewed the back translations and the first author made adjustments to the Korean versions (Behling & Law, 2000; Harkness, Van de Vijver et al., 2003).

The first author recruited Korean teachers from three groups: 13 peers of the first author from her undergraduate school, 32 teachers who were taking a qualification program[2] in eastern South Korea, and 21 teachers who were taking a graduate mathematics education class. The 96 U.S. high school teachers signed up voluntarily to participate in summer professional development projects taking place in two different states. They took the MMTsm as part of their professional development. U.S. and Korean teachers had similar years of teaching experience. The Korean teachers taught for an average of 4.5 years. This time included time teaching both middle school and high school mathematics. The 96 teachers in the U.S. sample analyzed in this paper taught at least one high school math class (algebra and above). We asked high school teachers how many times they had taught each subject and recorded the total number of high school classes taught. On average the U.S. teachers had taught 17.3 classes, which corresponds to approximately 4-5

[2] In Korea, all teachers who have taught more than three years must take a qualification training program to earn "1st class" teacher certificates.

years teaching. We also recorded the undergraduate major of teachers in the U.S. and Korea (See Table 1 and Table 2).

*Table 1. U.S. teachers' undergraduate majors.*

| | Math | MathEd | STE | Other | Total |
|---|---|---|---|---|---|
| Bachelor’s | 10 | 14 | 6 | 8 | 38 |
| Master’s | 17 | 22 | 5 | 14 | 58 |
| Total | 27 | 36 | 11 | 22 | 96 |

*Table 2. Korean teachers' undergraduate majors.*

| | Math | MathEd | Stat | Other | Total |
|---|---|---|---|---|---|
| Bachelor’s | 8 | 45 | 1 | 1 | 55 |
| Master’s | 1 | 10 | 0 | 0 | 11 |
| Total | 9 | 55 | 1 | 1 | 66 |

*Two Average Rate of Change Tasks*

Korean teachers saw 7 items on rate of change in MMTsm in the Summer of 2014. This report highlights the responses to two of these items. The item in Figure 1 is about a function’s average rate of change over an interval. One can answer this item’s question by joining two meanings: (1) an average rate of change is a constant rate of change, and therefore that it tells how many times as large a change in $y$ is as an associated change in $x$, and (2) that a difference between two values of a function is the amount the function changed between those two values.

Consider a non-linear function defined on the interval 7.3 to 7.6. The function’s average rate of change over that interval is 4. What is the difference between the value of the function at $x = 7.6$ and the value of the function at $x = 7.3$?

*Select the best answer.*

a. $0.3 \times 4$
b. 4
c. $0.3 / 4$
d. $4 / 0.3$
e. $7.6 - 7.3$
f. Not enough information.

Figure 1. *The item named "difference from rate."* © 2014 Arizona Board of Regents. Used with permission.

We constructed the multiple choice options from teachers’ answers to an earlier open-ended version and from teacher interviews. If the teacher thinks of average rate of change as “how many times as large a change in $y$ is as a change in $x$ over an interval”, we believe that they will select choice (a). During an interview one teacher explained that the answer is 4 because the interval of 0.3 as identical to an interval of 1. We suspect that teachers who picked a response with a quotient (choices (c) and (d)) thought that rate of change should involve a formula that has a quotient. In this sense, we hypothesized that if a teacher had a calculational approach such as "average rate of change is dividing change in $y$ by change in $x$" he would select (c) or (d). We included choice (e) to attract teachers who focused on the word "difference", so we think (e) reflects considering only change in $x$ values. Additionally, we anticipated that the teacher who believes the function must be linear to determine the answer would select (f).

We used the item "San Diego to El Centro" (Hackworth, 1994) to reveal teachers' meanings for the idea of average rate of change (see Figure 2).

| |
|---|
| A car went from San Diego to El Centro, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to San Diego if it were to have an average speed of 60 miles per hour over the round trip? |

Figure 2. *Part A of the item called "San Diego to El Centro".* © 2014 Arizona Board of Regents. Used with permission.

The item "San Diego to El Centro" is composed of two parts. Part A (shown in Figure 2) and Part B (whether the teacher's answer was consistent with the fact that it would take 3 hours at a constant speed to go 180 miles). We put Part B on a separate page to guard against teachers looking ahead to Part B before answering Part A.

We found this item to be particularly useful for revealing the meaning that an average rate of change is an arithmetic mean of rates. Teachers with this meaning typically solve the equation $\frac{40+x}{2}=60$, ending with an answer of 80 miles per hour. However, the desired meaning of average speed is the constant speed that the car would need to travel to go the same distance in the same amount of time as the actual trip. If the car were to travel 180 miles at a constant speed of 60 mi/hr, it would travel for 180/60 hours (3 hours). The car spent 2.25 hours traveling from San Diego to El Centro. It therefore has 0.75 hours remaining to travel the rest of the trip. So it must have an average speed of 120 mi/hr in the second leg of the trip to have an overall average speed of 60 mi/hr.

| |
|---|
| **Part B.** A round trip of 180 miles at an average speed of 60 mi/hr will take 3 hours. Is this fact consistent with your answer on the prior page? Explain.<br><br>If you would like to rework the problem, do so on this page. Please do not cross out your prior work. |

*Figure 3. Part B of the item called "San Diego to El Centro".* © 2014 Arizona Board of Regents. Used with permission.

We added Part B so as to see whether teachers can understand the inconsistency of an answer found from thinking of average speed as the arithmetic mean of two speeds. We added "please do not cross out your prior work" because, in earlier trials, some teachers crossed out their work on Part A after reading Part B. We focus later on whether Part B perturbed teachers' meanings for Part A.

Responses to "San Diego to El Centro" Part A were scored with a rubric. Responses to "San Diego to El Centro" Part B were scored in terms of whether the teacher thought the answer on Part A is consistent with the fact stated in Part B. The first author scored the Korean responses with the English rubric. The item “San Diego to El Centro” Part A was scored with the following rubric:

| Level | Level description | Sample response |
|---|---|---|
| **Level 3 Response:** | The response determined the return speed is 120mph by finding how much time remains for the second leg of the trip and computing the return speed accordingly. We ignore small computational errors if the response demonstrated a Level 3 type of reasoning. | SD, EC<br>$\frac{90\text{ mi}}{40\text{ m/hr}} = 2\frac{1}{4}$ hrs<br>90 miles<br>180 miles at 60 miles/hr. = 3hrs.<br>$\frac{90\text{ miles}}{3/4\text{ hr.}}$<br>$\cancel{90}^{30} \cdot \frac{4}{\cancel{3}} = \boxed{120\text{ miles/hr}}$ |
| **Level 2 Response:** | The response first wrote 80 mph, and then ultimately found a return speed of 120 mph. (*Note*: We believe teachers whose first instinct is incorrect are less likely to have strong meanings for average rate of change than a teacher who immediately uses a productive meaning for average rate of change.) | 120 ~~80~~ mi/hr<br>$\frac{90}{40}$ $2\frac{1}{4}$ hrs<br>$\frac{180\text{ mi}}{60\frac{\text{mi}}{\text{hr}}} = 3$ hrs<br>$\frac{90}{80}$<br>(0, 0)<br>(2.25, 90)<br>(3, 180)<br>90<br>60 mi/hr<br>S 90 E 90 S<br>$\frac{90}{3/4}$ |
| **Level 1 Response:** | Any of the following:<br>• The response found an arithmetic mean of two speeds (e.g. *(40 + S)/2 = 60*)<br>• The response found 80 mph without explicitly showing that they were using an arithmetic mean of two speeds. | SD → EC = 90mi at 40mph<br>$\frac{40+s}{2} = 60$<br>$40+s = 120$<br>$s = 80$mph |
| **Level 0 Response:** | Any of the following:<br>• I don’t know, scorer can’t interpret, work doesn’t address question.<br>• The response does not fit level 1 to 3 | $\frac{40}{90} = \frac{60}{180}$<br>I don't know how to do this accurately. I think it would be somewhere around 80-100 mph. |

We categorized responses that changed from an arithmetic mean of two speeds to a productive meaning for average speed at Level 2. As mentioned in the theoretical perspective, we focus on meanings that teachers might convey in classrooms. We imagine that students might construct mixed meanings for average rate of change (arithmetic mean and desired meaning) from the Level 2 teachers. In this sense, we think Level 2 responses are less productive than Level 3 responses.

We did not attend to computational errors when placing responses at Level 3. We only focused on whether the meanings we could discern from a teacher's response fit the item's purpose. In this sense, we ignored minor computational errors.

## Results

Teachers' responses to the item "difference from rate" are shown in Table 3.

*Table 3. Responses to "difference from rate."*

| Response | U.S. (any degree) | U.S. w/ Math Degree | U.S. w/ Math Ed. Degree | Korea |
|---|---|---|---|---|
| *0.3 * 4* | 47 | 14 | 18 | 61 |
| *4* | 9 | 1 | 6 | 1 |
| *0.3 / 4* | 12 | 2 | 6 | 0 |
| *4 / 0.3* | 11 | 3 | 3 | 1 |
| *7.6 - 7.3* | 6 | 2 | 2 | 0 |
| Not enough information | 19 | 3 | 1 | 3 |
| I don't know | 1 | 1 | 0 | 0 |
| No answer | 1 | 1 | 0 | 0 |
| Total | 96 | 27 | 36 | 66 |

About 49% of high school teachers from the United States gave the highest-level response, 0.3 times 4, to "difference from rate". About half of the teachers (51%) whose degree is mathematics or mathematics education from the United States gave the highest-level response. On the other hand, 61 out of 66 grade 7 to 12 teachers from Korea gave the highest-level response to this item. Since almost all Korean teachers had math or math education degree we did not distinguish the responses of Korean teachers by major.

The responses to "San Diego to El Centro" Part A show disparity between performance between U.S. and Korean teachers.

*Table 4. Responses to "San Diego to El Centro" Part A*

| Response | U.S. (any degree) | U.S. w/ Math Degree | U.S. w/ Math Ed. Degree | Korea |
|---|---|---|---|---|
| *Level 3* | 42 | 12 | 15 | 64 |
| *Level 2* | 8 | 3 | 4 | 0 |
| *Level 1 (80)* | 31 | 7 | 13 | 2 |
| *Level 0* | 14 | 4 | 4 | 0 |
| No answer | 1 | 1 | 0 | 0 |
| Total | 96 | 27 | 36 | 66 |

Approximately one third of teachers (31/96) in the United States revealed a meaning for average rate of change as an arithmetic mean of rates. Having a mathematics or mathematics education degree did not appear to be correlated to stronger meanings for average rate of change because 26% (7/27) of teachers with mathematics degree and 36% (13/36) of teachers with mathematics education degree showed a meaning for average rate of change as an arithmetic mean of rates. Only 2 out of 66 teachers in Korea had a meaning for average rate of change as an arithmetic mean of rates.

The responses to "San Diego to El Centro" Part B in U.S. show that most of the teachers in the United States that revealed a meaning for an arithmetic mean as average rate of change realized that it is not consistent with the fact given in Part B (Table 5).

*Table 5. United States Teachers Responses to "San Diego to El Centro" Part B*

| Response<br>U.S. Teachers | *Consistent* | *Not Consistent* | *Scorer Cannot Tell* | No answer | Total |
|---|---|---|---|---|---|
| *Level 3* | 41 | 0 | 0 | 1 | 42 |
| *Level 2* | 6 | 2 | 0 | 0 | 8 |
| *Level 1 (80)* | 5 | 22 | 4 | 0 | 31 |
| *Level 0* | 7 | 4 | 1 | 2 | 14 |
| No answer | 0 | 0 | 0 | 1 | 1 |
| Total | 59 | 28 | 5 | 4 | 96 |

Results from "San Diego to El Centro" Part B show that Part B perturbed teachers because about 71% of teachers who revealed a meaning for average rate of change as an arithmetic mean of rates wrote that their answer is not consistent with the fact that the trip will take 3 hours on Part B. However, five teachers who revealed a meaning for average rate of change as an arithmetic mean of rates stuck to the their original response of arithmetic mean of rates arguing that it is consistent with the given fact on Part B (Table 6).

*Table 6. Two sample responses in Level 1 and Consistent category*

| | Part A | Part B |
|---|---|---|
| Mr. Adams | SD El Centro.<br>90 miles at 40 m/hr.<br>speed = $\frac{\text{dist}}{\text{time}}$.<br>$\frac{40+x}{2} = 60$<br>$40 + x = 120$<br>$x = 80$<br>80 miles/hr. | This fact is consistent.<br>40 m/hr ~~1st~~ 1.5 hour = ~~40 miles~~ 60 miles<br>80 m/hr ~~2nd~~ 2nd 1.5 hour = ~~80 miles~~ ~~120 miles~~ $\frac{120}{180}$<br>Yes! |

| | | |
|---|---|---|
| Ms. Augusta | San Diego → El Centro<br>90 miles @ 40 miles/h<br>180 miles @ 60 miles/h<br>$\boxed{\text{80 miles/hr}}$ | yes - the average between 80 miles/hr. + 40 miles/h is 60 miles/hr. |

Both Mr. Adams and Ms. Augusta revealed an arithmetic mean of rates meaning for average rate of change. However, their responses to Part B show that Mr. Adams and Ms. Augusta's meanings for average rate of change are not identical. Mr. Adams applied the arithmetic mean of rates to the fact that the total trip will take 3 hours. He thought that the first leg of trip and the second leg of trip both took 1.5 hours. On the other hand, Ms. Augusta's response suggests that she did not check the fact that the total trip will take 3 hours. Rather, her response confirmed that "average" in average rate of change is no more than an arithmetic mean.

The responses to "San Diego to El Centro" Part B in Korea are in Table 7.

*Table 7. Korean Teachers Responses to "San Diego to El Centro" Part B.*

| Response | *Consistent* | *Not Consistent* | *Undecided* | No answer | Total |
|---|---|---|---|---|---|
| *Level 3* | 56 | 5 | 3 | 0 | 64 |
| *Level 2* | 0 | 0 | 0 | 0 | 0 |
| *Level 1 (80)* | 1 | 0 | 1 | 0 | 2 |
| *Level 0* | 0 | 0 | 0 | 0 | 0 |
| No answer | 0 | 0 | 0 | 0 | 0 |
| Total | 57 | 5 | 4 | 0 | 66 |

The reason why five Korean teachers in Level 3 wrote "not consistent" is that they made a computational error in Part A. Because we do not consider computational errors as part of our categorization system, the responses that show a Level 3 type of reasoning were categorized at Level 3. Three teachers in Level 3 wrote "I don't know", but we do not know why they wrote "I don't know". One possibility is that they did not carefully read Part B. One teacher in Level 1 stuck to an arithmetic mean as average rate of change in Part B, and the other in Level 1 wrote "I don't know".

## Conclusion

The results show that, in our convenience samples, Korean teachers' meanings for average rate of change are substantially stronger than U.S. teachers' meanings. Almost all Korean teachers knew that average rate of change tells us that a change in $y$ is a number of times as large as a change in $x$ over an interval and did not confound "average" with "arithmetic mean". We believe that the meanings teachers hold, such as average rate of change as an arithmetic mean, are the meanings they will operate with during instruction. It is likely that students will develop meanings for average rate of change that are similar to their teachers' meanings. Thus, if a teacher has incoherent meanings the probability is high that his students will develop incoherent meanings. Because average rate of change is a Common Core Mathematics Standard it is critical that teachers' have opportunities to learn this standard.

We could not investigate why teachers' in Korea have stronger meanings for average rate of change. However, some studies suggest a plausible possibility if we consider that Chinese, Hong Kong, and Korean students have similar performance on international tests. Ma (1999) identified that Chinese elementary teachers showed more profound understanding than U.S. elementary teachers even though U.S. teachers have longer formal schooling and higher degree. Leung (2006) also suggested that Hong Kong and Korean elementary teachers already acquired mathematics competence when they were students in school. Thus, we suspect that the disparity in U.S. and Korean teachers' meanings is because Korean teachers developed stronger meanings while students than did U.S. teachers while students. Put another way, teachers in Korea were students in Korea, and teachers in the U.S. were students in the U.S. It is possible that Korean teachers developed meanings as students that U.S. teachers did not develop. We agree with Stigler and Hiebert (1999) that teaching is a cultural activity and that teachers' experiences as students are highly influential in their later career as teachers.

Our results suggest that if school students are to develop strong meanings for average rate of change, pre-service teacher preparation programs in the U.S must ensure that their graduates develop strong meanings for average rate of change. We emphasize that a strong meaning for average rate of change involves other major ideas in the school curriculum. A strong meaning for average rate of change entails strong meanings for constant rate of change, which itself entails concepts of variation, covariation, and proportionality (Thompson & Thompson, 1996; Thompson, 1994a, 1994b; Thompson & Thompson, 1992, 1994).

One obvious limitation of our study is that the sample is not random and thus generalization of the results to the larger populations in either U.S. or Korea is not possible. However, the convenience samples did provide evidence that the Korean teachers understood the translated items in the way we intended and that the rubrics written based on grounded coding of U.S. teachers' responses were sufficient to categorize the range of Korean teacher's responses. The two items reported here required understandings of mathematics useful for teaching, but are not specifically related to student thinking. We anticipate that our analyses of rate of change items that asked Korean and U.S. teachers to respond to a teaching situation will allow us to speculate about teachers' attention to student thinking regarding average rate of change.

## Reference

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A*., et al.* (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*, 133-180.

Behling, O., & Law, K. S. (2000). *Translating questionnaires and other research instruments: Problems and solutions* (Vol. 133): Sage.

Bowers, J., & Doerr, H. M. (2001). An analysis of prospective teachers' dual roles in understanding the mathematics of change: Eliciting growth with technology. *Journal of Mathematics Teacher Education, 4*, 115-137. Retrieved from files/783/A1011488100551.html.

Byerley, C., & Thompson, P. W. (2014). Secondary teachers' relative size schemes. In P. Liljedahl & C. Nicol (Eds.), *Proceedings of the Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

Cai, J. (1995). A cognitive analysis of US and Chinese students' mathematical performance on tasks involving computation, simple problem solving, and complex problem solving. *Journal for Research in Mathematics Education. Monograph*, i-151.

Cho, S. (2010). *A Study on the Mathematical Knowledge for Teaching (MKT) in Differentiation for Second grade in High School.* Master's thesis, The Graduate School of Education Ewha Womans University, Seoul.

Coe, E. E. (2007). *Modeling Teachers' Ways of Thinking About Rate of Change.* Arizona State University. Available from Google Scholar

Corbin, J., & Strauss, A. (2007). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.): Sage Publications, Inc. Retrieved from http://www.amazon.com/dp/141290644X.

Hackworth, J. A. (1994). *Calculus students' understanding of rate.* Masters Thesis, San Diego State University, Department of Mathematical Sciences. Retrieved from http://pat-thompson.net/PDFversions/1994Hackworth.pdf

Hanushek, E. A., Peterson, P. E., & Woessmann, L. (2010). *US Math Performance in Global Perspective: How Well Does Each State Do at Producing High-Achieving Students? PEPG Report No.: 10-19.*: Harvard University

Harkness, J. A., Van de Vijver, F. J., Mohler, P. P., & fur Umfragen, Z. (2003). *Cross-cultural survey methods* (Vol. 325): Wiley-Interscience Hoboken.

Hill, H. C., Ball, D. L., Blunk, M., Goffney, I. M., & Rowan, B. (2007). Validating the ecological assumption: The relationship of measure scores to classroom teaching and student learning. *Measurement, 5*, 107-118.

Kim, Y.-O. (2007). *Middle school mathematics teachers' subject matter knowledge for teaching in China and Korea*: ProQuest.

Leung, K. S. F. (2006). Mathematics education in East Asia and the West: Does culture matter? In *Mathematics education in different cultural traditions-a comparative study of east asia and the west* (pp. 21-46): Springer.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Skemp, R. R. (1962). The need for a schematic learning theory. *The British Journal of Educational Psychology, 32*, 133–142.

Skemp, R. R. (1971). *The psychology of learning mathematics*. [Harmondsworth, Eng., Baltimore]: Penguin Books.

Steffe, L. P. (1991). The learning paradox. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26–44). New York: Springer-Verlag.

Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom* (Vol. 1st (updated)). New York: Free Pess.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal, 11*, 124-144.

Tatto, M. T., Ingvarson, L., Schwille, J., Peck, R., Senk, S. L., & Rowley, G. (2008). *Teacher Education and Development Study in Mathematics (TEDS-M): Policy, Practice, and*

*Readiness to Teach Primary and Secondary Mathematics. Conceptual Framework*: ERIC.

Tatto, M. T., Peck, R., Schwille, J., Bankov, K., Senk, S. L., Rodriguez, M., *et al.* (2012). *Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-MM)*: ERIC.

Tatto, M. T., & Senk, S. (2011). The mathematics education of future primary and secondary teachers: Methods and findings from the Teacher Education and Development Study in Mathematics. *Journal of Teacher Education, 62*, 121-137.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*, 2-24.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*, 229–274.

Thompson, P. W. (2013). In the absence of meaning…. In *Vital directions for mathematics education research* (pp. 57-93): Springer.

Thompson, P. W. (2015). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third handbook of international research in mathematics education* (pp. 968-1002). New York: Taylor & Francis.

Thompson, P. W. (in press). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third Handbook of International Research in Mathematics Education*. London: Taylor and Francis.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., & Thompson, A. G. (1992). *Images of rate*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA. Retrieved from http://bit.ly/17oJe30

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*, 279–303.

Weber, E., & Dorko, A. (2014). Students' and experts' schemes for rate of change and its representations. *The Journal of Mathematical Behavior, 34*, 14-32.

Yoon, H., Hatfield, N., & Thompson, P. W. (2014). Teachers' meanings for function notation. In P. Liljedahl & C. Nicol (Eds.), *Proceedings of the Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

-