MODULE - I

# 1- HARMONIC OSCILLATIONS



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- Simple Harmonic Motion
  - → Equation & Solution
- Damped Harmonic Motion
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- Forced Harmonic Motion
  - → Equation & Solution
- Resonance
  - → Sharpness of Resonance
- Quality Factor
- LCR circuit

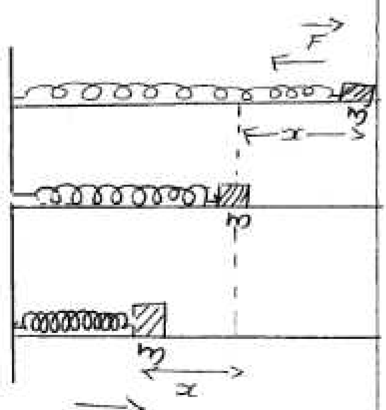
## Simple Harmonic Motion (SHM)

A motion that repeated at regular intervals of time is called periodic motion. Simple Harmonic Motion is a type of periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

eg:- Oscillations of simple pendulum
Loaded spring

- Equation of Motion

When spring is stretched a distance '$x$', then the restoring force acting on the mass is




$$F \propto -x \,,$$

$F \rightarrow$ Restoring force, $k \rightarrow$ constant of proportionality

$$\boxed{F = -kx}$$

-ve sign indicates force $F$ & $x$ are in oppo

In $F = -kx$, the $-ve$ sign indicates that the restoring force and the displacement are in opposite direction.

ie Acceleration $= \frac{d^2x}{dt^2}$

ie $m\frac{d^2x}{dt^2} = -kx$

or $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$ ; where $\omega_0 = \sqrt{k/m}$ — 1

This is the D.E. of SHM.

The solution of 1 is given by

$x = a \sin(\omega_0 t + \phi)$

$x \rightarrow$ displacement
$a \rightarrow$ amplitude
$\omega \rightarrow$ angular frequency
$\phi \rightarrow$ phase angle
$\omega_0 t + \phi \rightarrow$ phase

frequency $f = \frac{1}{T}$
$\omega_0 = 2\pi f$

velocity of the particle

$v = \omega_0 \sqrt{a^2 - x^2}$

## Damped Harmonic Oscillations

It is a SHM in which amplitude is steadily decreased due to the action of damping force like friction or air resistance.

Damping force $\propto$ velocity

ie $f = -b\frac{dx}{dt}$

$b \rightarrow$ damping constant

- Equation of motion for DHO

Here

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\therefore \boxed{\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = 0} \quad \text{--- (2)}$$

Let $k/m = \omega_0^2$
$b/m = 2r$

This is the D.E of DHO

Solution:

Assume the solution (2) is

$$x = Ae^{\alpha t}$$

So $\frac{dx}{dt} = A\alpha e^{\alpha t}$

& $\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$

$\therefore$ (2) $\Rightarrow$ $\alpha^2 x + 2r\alpha x + \omega_0^2 x = 0$

ie $\alpha^2 + 2r\alpha + \omega_0^2 = 0$

The root of this equation is

$$\alpha = \frac{-2r \pm \sqrt{4r^2 - 4\omega_0^2}}{2}$$

$$\therefore \boxed{x = A_1 e^{(-r+\sqrt{r^2-\omega_0^2})t} + A_2 e^{(-r-\sqrt{r^2-\omega_0^2})t}} \quad \text{--- (3)}$$

Where $A_1$ & $A_2$ are constants.

Classification

1- Underdamped (low damped) => oscillatory but amplitude decreases with time
2- Critically damped => Not oscillatory. (small damping)
3- Over damped => Not Oscillatory (High damping)

## Case I : Underdamped case or low damped case

The condition is $r < \omega_0$

$\because r < \omega_0 \Rightarrow \sqrt{r^2 - \omega_0^2} = i\sqrt{\omega_0^2 - r^2}$

$$\therefore x = e^{-rt}\{A_1 e^{i\omega t} + A_2 e^{-i\omega t}\}$$

$$= e^{-rt}\{\cos\omega t\,(A_1 + A_2) + i\sin\omega t\,(A_1 - A_2)\}$$

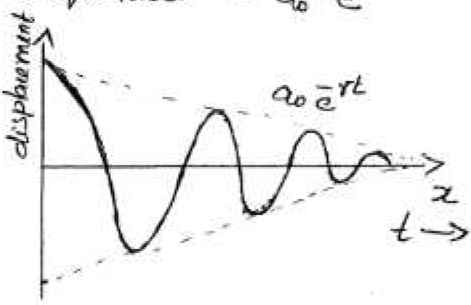
Let $A_1 + A_2 = a_0 \sin\phi$

$i(A_1 - A_2) = a_0 \cos\phi$

$$x = e^{-rt}\{a_0 \sin\phi \cos\omega t + a_0 \cos\phi \sin\omega t\}$$

$\boxed{x = a_0 e^{-rt} \sin(\omega t + \phi)}$ $\Rightarrow$ This motion is oscillatory.

here amplitude $= a_0 e^{-rt}$




$\Rightarrow$ Damped oscillation (under damped)

## Case II : Critically damped

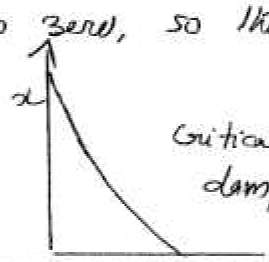
The condition is $r = \omega_0$

Applying $r = \omega_0$ in eq (3)

$$x = A_1 e^{-rt} + A_2 e^{-rt}$$

$$= C e^{-rt} \qquad \text{where } C = A_1 + A_2$$

Here the displacement decreases to zero, so this is not an oscillatory motion




Critically damped.

Case III : over damped ( High damping )

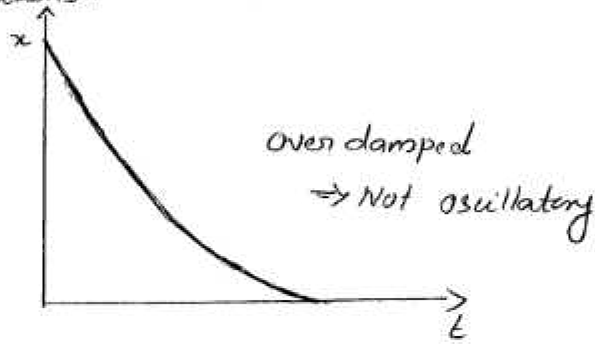
The condition is $\Gamma > \omega_0$

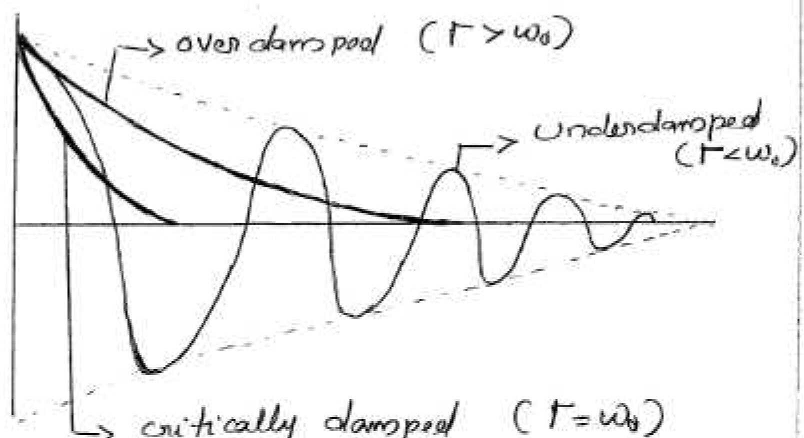


$\because \Gamma > \omega_0$ ; $\sqrt{\Gamma^2 - \omega_0^2} < \Gamma$

$\therefore \left(-\Gamma + \sqrt{\Gamma^2 - \omega_0^2}\right)t$ and $\left(-\Gamma - \sqrt{\Gamma^2 - \omega_0^2}\right)t$ are negative

So the displacement ($x$) decreases exponentially to zero without any oscillations.




Forced Harmonic Oscillation ( Driven oscillations )

If an external periodic force is applied on a damped harmonic oscillator, then the oscillating system is called forced harmonic oscillator and its oscillation is called forced harmonic oscillations

When a body is executing oscillations under an external force; three forces are acting on it

1. A restoring force which is proportional to the displacement & oppositively directed
2. A damping force which is proportional to the velocity & oppositively directed
3. An external periodic force is represented as $f_0 \sin \omega_f t$

The eqn of motion of FHO is:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + f_0 \sin\omega_f t$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f_0 \sin\omega_f t$$

$$\boxed{\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f_0 \sin\omega_f t} \quad (4)$$

$\frac{b}{m} = 2r$

$\frac{k}{m} = \omega_0^2$

$\frac{F_0}{m} = f_0$.

At first the body oscillates neither with the natural frequency ($\omega_0$) nor with the frequency of applied force. But after some time the system reaches at a steady state and the oscillator oscillates with the frequency of the external periodic force.

Let us assume the solution of (4) is

$$x = A \sin(\omega_f t - \theta)$$

so $$\frac{dx}{dt} = A\omega_f \cos(\omega_f t - \theta)$$

$$\frac{d^2x}{dt^2} = -A\omega_f^2 \sin(\omega_f t - \theta)$$

$$\therefore (4) \Rightarrow -A\omega_f^2 \sin(\omega_f t - \theta) + 2r A\omega_f \cos(\omega_f t - \theta)$$
$$+ \omega_0^2 A \sin(\omega_f t - \theta) = f_0 \sin(\omega_f t - \theta + \theta)$$

$$-A\omega_f^2 \sin(\omega_f t - \theta) + 2r A\omega_f \cos(\omega_f t - \theta) + \omega_0^2 A \times$$
$$\sin(\omega_f t - \theta) = f_0\left[\sin(\omega_f t - \theta)\cos\theta + \cos(\omega_f t - \theta) \times \sin\theta\right]$$

$$\Rightarrow \sin(\omega_f t - \theta)\left[-A\omega_f^2 - f_0\cos\theta + A\omega_0^2\right] +$$
$$\cos(\omega_f t - \theta)\left[2rA\omega_f - f_0 \sin\theta\right] = 0$$

To find $A$,

The coefficient $\sin(\omega_f t - \theta)$ & $\cos(\omega_f t - \theta)$ are separately equal to zero

ie
$$-A\omega_f^2 - f_0\cos\theta + A\omega_0^2 = 0$$
$$\& \quad 2rA\omega_f - f_0\sin\theta = 0$$

OR
$$A\omega_0^2 - A\omega_f^2 = f_0\cos\theta \quad — (5)$$
$$2rA\omega_f = f_0\sin\theta \quad — (6)$$

Squaring and adding (ie $(5)^2 + (6)^2$)

$$(A\omega_0^2 - A\omega_f^2)^2 + (2rA\omega_f)^2 = f_0^2$$

$$\therefore A^2 = \frac{f_0^2}{(\omega_0^2 - \omega_f^2)^2 + 4r^2\omega_f^2}$$

$$\therefore A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + 4r^2\omega_f^2}}$$

ie The amplitude of forced harmonic oscillator is

$$\boxed{A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + 4r^2\omega_f^2}}} \quad — (7)$$

Dividing ⑥ & ⑤ (i.e ⑥/⑤)

$$\boxed{\tan\theta = \frac{2r\omega_f}{\omega_0^2 - \omega_f^2}} \; ; \quad \theta = \tan^{-1}\left[\frac{2r\omega_f}{\omega_0^2 - \omega_f^2}\right]$$

This is the phase difference between the forced oscillations and applied force.

Therefore; the solution of eq ④ is given by

$$x = \frac{f_0}{\sqrt{(\omega_0^2-\omega_f^2)^2 + 4r^2\omega_f^2}} \sin(\omega_f t - \theta) \; ; \quad \text{where } \theta = \tan^{-1}\left[\frac{2r\omega_f}{\omega_0^2-\omega_f^2}\right]$$

Case I : Low driving frequency:

$\omega_f < \omega_0$

we have $A = \dfrac{f_0}{\sqrt{(\omega_0^2-\omega_f^2)^2 + 4r^2\omega_f^2}}$

$\because \omega_f < \omega_0$ ; neglecting $\omega_f^2$

$$\therefore A = \frac{f_0}{\omega_0} = \frac{F_0/m}{k/m}$$

$A = \dfrac{F_0}{k}$ $\Rightarrow$ Amplitude depends on force constant but it doesnot depend on mass of the body.

Case II : $\omega_f > \omega_0$ High driving frequency.

when $\omega_f > \omega_0$ ; $\omega_0^2$ can be neglected

so $A = \dfrac{f_0}{\sqrt{(\omega_f^2)^2 + 4r^2\omega_f^2}}$

for low damping; $r^2$ can be neglected

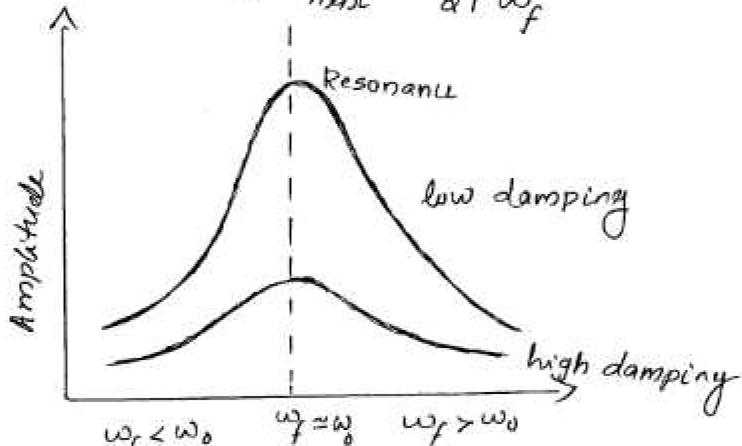
i.e $A = \dfrac{f_0}{\omega_f^2}$

## Case III : Resonance

$\omega_f \simeq \omega_0$

When $\omega_f$ is increasing and $r$ is small, then the amplitude of the oscillations increases, and reaches a maximum value when both frequencies are nearly equal ($\omega_f \simeq \omega_0$). This phenomenon is called resonance; and corresponding value of frequency is called resonant frequency. At resonance amplitude is maximum.

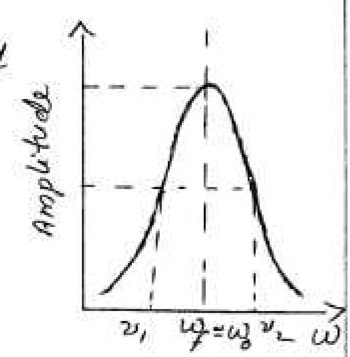
So at resonance; $A_{max} = \frac{f_0}{2r\omega_f}$




This is the variation of amplitude with periodic frequency $\omega_f$

## Sharpness of resonance:

The sharpness of resonance is defined as the rate of decrease of amplitude with the change in frequency of the applied periodic force on either side of the resonant frequency.




Band width of the oscillator: At resonance the power absorbed is maximum. $\nu_1$ & $\nu_2$ are the frequencies corresponding to half power point ( ie the power absorbed is half of the maximum value).

The difference in frequency between these two half power point is called the band width of oscillator

$\Delta\nu = \nu_2 - \nu_1$

## Quality factor ( Q)

Quality factor defines the quality or efficiency of the oscillator

- High Q value indicates, a lower rate of energy loss.
- Quality factor or (Q- factor) is defined as 2π times the energy stored to the energy dissipated per cycle.

ie $$Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy dissipated per cycle}}$$

we have

$$x = a_0 e^{-rt} \sin(\omega t + \phi)$$

Let '$E_t$' be the energy stored at time 't' sec

we have $a = a_0 e^{-rt}$

Energy ∝ Amplitude$^2$

$$E_t = a_0^2 e^{-2rt}$$

Energy stored after a time 'T' sec

$$E_{t+T} = a_0^2 e^{-2r(t+T)} = a_0^2 e^{-2rt} \cdot e^{-2rT}$$
$$= E_t e^{-2rT}$$

∴ Energy loss per cycle $= E_t - E_{t+T} = E_t(1 - e^{-2rT})$

$$= E_t [1 - (1 - 2rT)]$$
$$= 2rT E_t.$$

$$Q = 2\pi \frac{E_t}{E_t \cdot 2r\,T} = \frac{2\pi}{2r\,T} = \frac{\omega_0}{2r}$$

OR

Q-factor is also defined as the ratio of amplitude at resonance to the amplitude at zero driven frequency.

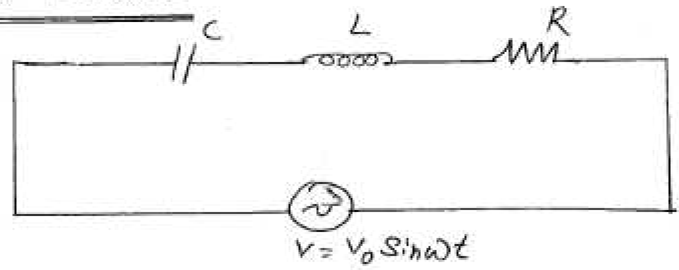
ie $$Q = \frac{\text{Amplitude at resonance}}{\text{Amplitude at zero driven frequency}}$$

Amplitude at resonance ; $A = \dfrac{f_0}{2r\omega_0}$ $(\because \omega = \omega_0)$

Amplitude at zero driven frequency; $A = \dfrac{f_0}{\omega_0^2}$

$$\therefore Q = \frac{f_0 / 2r\omega_0}{f_0 / \omega_0^2} = \frac{\omega_0^2}{2r} = \frac{\omega_0^2}{2r}$$

LCR circuit




LCR circuit is an example of both Damped H.O & Forced H.O.

- If Q be the charge in the conductor,
  C be the capacitance

Thus $V = Q/C$

- Let $I = \frac{dQ}{dt}$ is the current in the circuit, thus induced emf in the inductance $= -L\frac{dI}{dt}$
- According to ohm's law, the potential difference across the resistor $= IR$

Total potential difference = total emf

$$IR + Q/C = -L\frac{dI}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

or $$\boxed{\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0}$$

This differential equation is similar to the equation of damped H.O.

Comparing; $2r = R/L$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

solution is

$$Q = Q_0 e^{-\frac{R}{2L}t}\sin(\omega t + \phi)$$

If an ac emf $V = V_0 \sin\omega t$ is applied to the LCR circuit; then

$$\boxed{\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{V_0}{L}\sin\omega t.}$$

solution is

$$Q = \frac{V_0/L}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{\omega R}{L}\right)^2}}$$

Equivalence between the mechanical oscillator and electrical oscillator.

| Mechanical oscillator | Electrical oscillator. |
|---|---|
| 1. Mass (m) | Inductance (L) |
| 2. Displacement (x) | Charge (Q) |
| 3. Velocity - $dx/dt$ | Current $I = \frac{dQ}{dt}$ |
| 4. Damping Coefficient (r) | Electrical resistance (R) |

| Mechanical Oscillator | Electrical oscillator |
|---|---|
| 5- force constant ($k$) | Reciprocal of capacitance ($1/C$) |
| 6- Resonant angular frequency $\omega_0 = \sqrt{k/m}$ <br> So Resonant frequency $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ | Resonant angular frequency in the case of LCR circuit $\omega_0 = \frac{1}{\sqrt{LC}}$ <br> Resonant frequency $\nu_0 = \frac{1}{2\pi\sqrt{LC}}$ |
| 7- Q-factor $= \frac{\omega_0}{2r}$ $= \frac{\sqrt{k/m}}{2r}$ | Q-factor $= \frac{1}{2R}\sqrt{\frac{L}{C}}$ |
| 8. Potential energy $= \frac{1}{2}kx^2$ | Energy stored in capacitor $= \frac{q^2}{2C} = \frac{1}{2}CV^2$ |
| 9. Kinetic energy $= \frac{1}{2}mv^2$ | Energy stored in inductor $= \frac{1}{2}LI^2$ |
| 10- Differential equation $\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = 0$ | Differential equation $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$ |

Questions from KTU Question Paper.

1- What do you mean by quality factor of an oscillator
[2 marks - January/2016]

2- Compare an electrical and Mechanical oscillator
[4 marks - January/2016]

3- Distinguish between free oscillation and damped oscillation
[2 marks - May/2016]

4. What are the conditions for oscillations of a harmonic oscillator to be over damped, critically damped, and under damped. Compare the time-displacement curve in the three cases [4 marks: May/2016]

5. Write the differential equation of forced harmonic oscillator and write its solution. Derive the expression for the amplitude and phase difference in terms of the natural frequency of the body and the frequency of the applied periodic force.

Problems:

1. For a damped oscillator, the mass of the block is 200 gm, force constant $= 10\,N m^{-1}$, and damping constant is 40 g/s. Examine whether the motion is oscillatory or not; if oscillatory find the period.

Condition for oscillatory; $r < \omega_0$

$b = 40\ g/sec$ $\quad r = \frac{b}{2m}$ & $\omega_0 = \sqrt{k/m}$

$m = 200\ g$

$k = 10\ N/m$

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - r^2}}$$

2. The differential equation of SHM is given by $\frac{d^2y}{dt^2} + 100y = 0$. Find the period and the frequency of SHM

3. The equation of motion of particle is given by $x = 2\sin(20t + \pi/2)$. Find the period and frequency and also find the maximum velocity.

$$\left\{ \begin{array}{l} v = \omega\sqrt{a^2 - x^2} \\ v_{max} = a\omega \end{array} \right\}$$

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# MODULE - I 2: WAVES

## Wave:

- It is a disturbance produced in an elastic medium due to the periodic vibrations of particles.
- It is the transfer of energy from one point to another point.
- There are two types of wave motion.

1. Transverse waves 2. Longitudinal waves

## Transverse wave:

When the particles of the medium vibrates in a direction perpendicular to the direction of propagation of the waves is called transverse waves.

eg:- Electromagnetic waves
Ripples on the surface of water

## Longitudinal wave:

When the particle of the medium vibrates in a direction parallel to the direction of propagation of the wave, is called longitudinal wave

eg:- Sound waves

## General form of travelling wave.

Consider a transverse wave pulse travelling in a positive 'x' direction

Let $U(x,t)$ be the displacement

at $t=0$ ; $U(x,0) = F(x)$ —— (1)

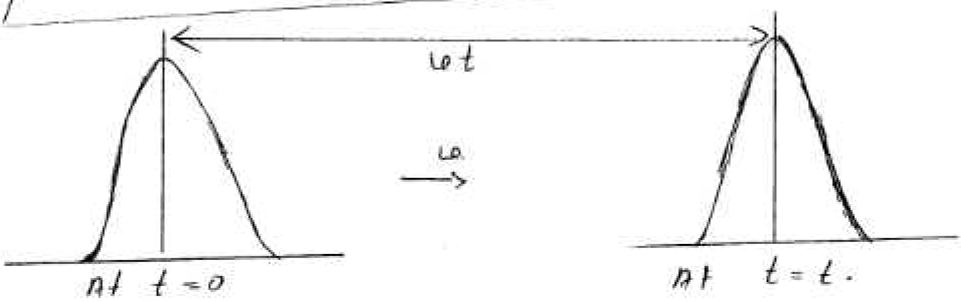
After a time 't' sec; the wave has travelled a distance $vt$ with $v \to$ velocity of the wave

Then at 't'; $U(x,t) = f(x-vt)$ —— ②
(wave moving in +ve x dirn)

If the wave moving in −ve x direction;
$U(x,t) = f(x+vt)$

∴ General form is

$$\boxed{U(x,t) = f_1(x+vt) + f_2(x-vt)}$$



One dimensional wave equation:

Consider a wave travelling in +ve x direction

Then displacement

$U(x,t) = f(x-vt)$ —— ③

$\frac{\partial U}{\partial x} = f'(x-vt)$ & $\frac{\partial U}{\partial t} = -v f'(x-vt)$

$\frac{\partial^2 U}{\partial x^2} = f''(x-vt)$ —— ④ & $\frac{\partial^2 U}{\partial t^2} = v^2 f''(x-vt)$ —— ⑤

⑤ $\Rightarrow \frac{\partial^2 U}{\partial t^2} = v^2 \frac{\partial^2 U}{\partial x^2}$ (from ④)

ie $\boxed{\frac{\partial^2 U}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}}$ This is the differential equation of one-D wave motion

$\frac{\partial U}{\partial t} \to$ Particle velocity. (one-D wave equat

Solution

we have $\frac{\partial^2 U}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}$ — 6

Let the solution will be

$U_{(x,t)} = X_{(x)} T_{(t)}$ — 7

$X(x) \rightarrow$ function of position, x
$T(t) \rightarrow$ function of time, t

Sub 7 in 6

$$T\frac{d^2X}{dx^2} = \frac{1}{v^2} X \frac{d^2T}{dt^2}$$

Divided by XT on both sides

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Tv^2}\frac{d^2T}{dt^2}$$

Here LHS is a function of x only and RHS is a function of t only.

Hence each side must be equal to a constant $-k^2$ {∵ change in x will not change the right side and change in 't' will not change the left side}.

ie $\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \Rightarrow \frac{d^2X}{dx^2} + k^2X = 0$ — 8

& $\frac{1}{Tv^2}\frac{d^2T}{dt^2} = -k^2 \Rightarrow \frac{d^2T}{dt^2} + k^2v^2T = 0$ — 9

$\therefore \underline{k^2v^2 = \omega^2}$ $\quad\therefore$ 9 $\Rightarrow \frac{d^2T}{dt^2} + \omega^2T = 0$ — 10

8 and 10 are the standard differential equation with solution is given by

$X_{(x)}$ = constant $e^{\pm ikx}$

$T(t)$ = constant $e^{\pm i\omega t}$

∴ The solution is $U_{(x,t)} = A\ \exp(i(kx \pm \omega t))$

$$\boxed{U_{(x,t)} = A e^{i(kx \pm \omega t)}}$$

This is the solution of one-D wave equation.

$$U(x,t) = A \sin(kx - \omega t)$$
$$= A\sin\left(\frac{2\pi}{\lambda}x - 2\pi\nu t\right)$$
$$U(x,t) = A\sin\frac{2\pi}{\lambda}(x - vt)$$

$\because k = \frac{2\pi}{\lambda}$

$\omega = 2\pi\nu$

$\nu = \frac{1}{T}$

$v = \nu\lambda$

## Three Dimensional Wave equation and its Solution:

If the wave propagating in any direction, then the D.E is given by

$$\boxed{\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}} \quad \text{--- (11)}$$

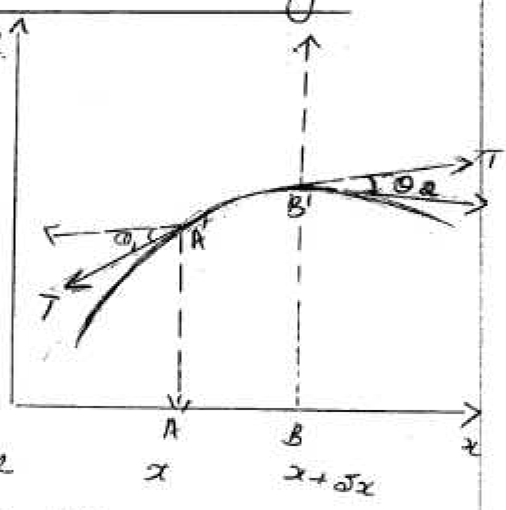
or

$$\nabla^2 U = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}$$

Solution of (11) is

$$\boxed{U(x,y,z) = A\, e^{i(\vec{k}\cdot\vec{r} \pm \omega t)}}$$

## Transverse Vibration in a Stretched String:




Consider an infinitely long, thin, Uniform string stretched between A & B by a constant tension T.

Let the string be slightly displaced along the y-axis and released, then transverse vibrations are formed in the string.

- Let a small element AB of length $= \delta x$
- Magnitude of the tension will be same every where ($\because$ the string is perfectly flexible)
- Tension T acts tangentially at every point
- At A', tension T makes an angle $\theta_1$ & at B', tension T makes an angle $\theta_2$ with horizontal.

The net force acting along x & y direction is

$$F_x = T\cos\theta_2 - T\cos\theta_1 \quad \text{--- (12)}$$
$$F_y = T\sin\theta_2 - T\sin\theta_1 \quad \text{--- (13)}$$

for small oscillations, $\theta_1$ & $\theta_2$ are very small

so $\cos\theta_1 \simeq \cos\theta_2 \simeq 1$

& $\sin\theta_1 \simeq \tan\theta_1$ & $\sin\theta_2 \simeq \tan\theta_2$

$\therefore$ (12) $\Rightarrow$ $F_x = 0$

& $F_y = T[\tan\theta_2 - \tan\theta_1]$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right] \quad \text{--- (14)}$$

from Newton's second law of motion,

$$F = ma$$
$$= \mu\,\delta x \frac{\partial^2 y}{\partial t^2} \quad \text{--- (15)}$$

$a = \frac{\partial^2 y}{\partial t^2}$

$\mu \rightarrow$ linear mass density (mass/unit length)

$\therefore$ mass of the string AB $= \mu\,\delta x$.

$\therefore$ from (14) & (15)

$$T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right] = \mu\,\delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right]}{\delta x} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

In the lim $\delta x \to 0$; $\dfrac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x} = \dfrac{\partial^2 y}{\partial x^2}$

Therefore the above equation becomes

$$\therefore \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}} \quad \text{--- (16)}$$

This is the wave equation (D.E) in the case of waves in a stretched string.

Comparing (16) with $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

we get $v^2 = T/\mu$

or velocity of transverse wave on a stretched string is $\boxed{v = \sqrt{T/\mu}}$

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Questions from KTU Question Paper.

1- A transverse wave on a stretched string is described by $Y(x,t) = 4.0 \sin(25t + 0.016x + \pi/3)$ where $x$ & $y$ are in cm & $t$ is in second. Obtain (i) speed (ii) Amplitude (iii) Frequency (iv) Initial phase at origin.

[4 Marks; January/2016]

2- Considering the transverse vibrations in a stretched string, derive the differential equation of one dimensional wave. [6 Marks; January/2016]

3- State the laws of transverse vibrations of a stretched string. [2 Marks; May-2016]

Answer:

Laws of transverse vibration of a string

1- Law of length:

The fundamental frequency of vibration of a string (fixed at both ends) is inversely proportional to the length of the string provided its tension and mass/unit length remain the same.

$\nu \propto \frac{1}{l}$ ; If $T$ & $\mu$ are constant

2- Law of tension:

The fundamental frequency of a string is proportional to the square root of its tension provided its length and mass per unit length remain the same

$\nu \propto \sqrt{T}$ ; if $l$ and $\mu$ are constant

3- Law of mass:

The fundamental frequency of string is inversely proportional to the square root of the linear mass density provided the length and the tension remain the same.

$\nu \propto \frac{1}{\sqrt{\mu}}$ ; If $l$ and $T$ are constant

$\boxed{\nu = \frac{1}{2l}\sqrt{\frac{T}{\mu}}}$ => Fundamental frequency (1st harmonic)

4. A piece of wire 50 cm long is stretched by a load of 2.5 kg and has a mass of 1.44 g. Find the frequency of the second harmonic

[4 Marks ; May /2016]

frequency of nth harmonic

$$f_n = \frac{n}{2l}\sqrt{T/\mu}$$

frequency of second harmonic

$$f_2 = \frac{2}{2l}\sqrt{T/\mu}$$

$$= \frac{1}{l}\sqrt{T/\mu}$$

$m = 1.44g$

$l = 50cm$

$\mu$ = mass / length $= \frac{1.44 \times 10^{-3} kg}{50 \times 10^{-2} m} = 0.0288 \times 10^{-1}$

$= 0.288 kg/m$

Tension $T = mg$

$= 2.5 kg \times 9.8$

$= 24.5 kg m/s^2$

$\therefore f = \frac{1}{50 \times 10^{-2}}\sqrt{\frac{24.5}{0.288}} =$

Problems.

1. Calculate the speed of transverse wave in a string of cross sectional area $1 mm^2$ Under tension of 1kg.wt. Density of Wire $= 10.5 \times 10^3 kg/m^3$

$\mu$ = mass / Unit length

density $= \frac{m}{volume}$

$\mu$ = area x density

$= \pi r^2 \times$ density.

$T = 1 kg wt = 1 \times 9.8 N$