Curriculum Burst 34: A Third of an Input

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> Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$.
>
> Find the sum of all values z for which $f(3z) = 7$.

SOURCE: This is question # 15 from the 2000 MAA AMC 12 Competition.

QUICK STATS:

YouTube

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MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Algebra: Solving Quadratics

COMMON CORE STATE STANDARDS

A-SSE.3a: Factor a quadratic expression to reveal the zeros of the function it defines.
A-SSE.3b: Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: DO SOMETHING

THE PROBLEM-SOLVING PROCESS:

As always, the appropriate first step…

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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This question strikes me as a little odd. For an input x we are given information about $f(x/3)$, the output for one third of that input. Weird!

We actually have a formula: $f\left(\frac{x}{3}\right)=x^2+x+1$. Hmm.

Just a to get a feel for things, can I work out $f(20)$? (I don't know why I chose 20. I am just trying something.)

Well, 20 is a third of 60, so

$$f(20)=f\left(\frac{60}{3}\right)=60^2+60+1$$

which I can work out if I wanted to.

Actually, this shows me what to do in general. To work out $f(N)$ for some input N, think of N as a third of another number and then use the formula x^2+x+1 for that number. For instance, $f(2)$ is 6^2+6+1, and $f(.11)$ is $.33^2+.33+1$.

Okay. What was the question?

We want to find all the values z for which $f(3z)=7$.

Well, $3z$ is a third of $9z$, so we want all values for which $(9z)^2+(9z)+1=7$. That is, we hope to solve:

$$81z^2+9z+1=7.$$

One can use the quadratic formula, I suppose, but I don't have it in my head. Allow me to complete the square … literally! (See www.gdaymath.com/courses/quadratics-2-the-algebra-of-quadratics for an explanation of this approach. What I choose to do next might seem strange if you are not familiar with his natural idea.)

The odd coefficient of 9 in the middle of the equation is awkward. Let's multiply everything through by four. (This keeps the first term a perfect square.)

$$324z^2+36z+4=28$$

	18z	**1**
18z	**324 z²**	**18z**
1	**18z**	**1**

To complete the square we see we need the number 1, not 4, in the left. Let's subtract three from each side:

$$324z^2+36z+1=25.$$

The picture now makes it clear we have:

$$(18z+1)^2=25$$
$$18z+1=5 \quad or \quad -5$$
$$18z=4 \quad or \quad -6$$
$$z=\frac{2}{9} \quad or \quad -\frac{1}{3}$$

This does it! Actually, the question asks for the sum of possible values of z. This sum is $\frac{2}{9}+\left(-\frac{3}{9}\right)=-\frac{1}{9}$. Now we are done!

Extension: In solving $f(3z)=7$, both solutions for z were fractions. Show that in solving $f(3z)=91$, one solution for z is an integer (but not the other).

CHALLENGE: Prove that there is no positive integer k for which both solutions for z in $f(3z)=k$ are integers!

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