International Journal of Applied Mathematical Sciences
ISSN 0973-0176 Volume 8, Number 1 (2015), pp. 31-54

http://www.ripublication.com

A New Transmuted Additive Weibull Distribution: Based On A New Method For Adding A Parameter To A Family Of Distributions

Mahmoud M.Mansour[1], Enayat M. Abd Elrazik[2], Mohamed S. Hamed[3] and Salah M. Mohamed[4]

[1]*Dept. of Statistics, Mathematics and Insurance, Benha University, Egypt*
E-mail: *mahmoud.mansour@fcom.bu.edu.eg*
[2]*Dept. of Statistics, Mathematics and Insurance, Benha University, Egypt*
E-mail: *anayat.khalil@ fcom.bu.edu.eg*
[3]*Dept. of Statistics, Mathematics and Insurance, Benha University, Egypt*
E-mail: moswilem@gmail.com
[4]*Institute of Statistical Studies & Research, Cairo University, Egypt*
E-mail: dr.mahdym62@gmail.com

Abstract

This paper introduces a new generalization of the transmuted additive Weibull distribution by Elbatal and Aryal [10], based on a new family of lifetime distribution. We refer to the new distribution as a new transmuted additive Weibull (NTAW) distribution. The new model contains some of lifetime distributions as special cases such as the transmuted additive Weibull, exponentiated modified Weibull, exponentiated Weibull, exponentiated exponential, transmuted Weibull, Rayleigh, linear failure rate and exponential distributions, among others. The properties of the new model are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions are derived for the moments and examine the order statistics. An application to real data set is finally presented for illustration.

Keywords: transmutation; survival function; exponentiated exponential; order statistics; maximum likelihood estimation.

Introduction

For complex electronic and mechanical systems, the failure rate often exhibits non-monotonic (bathtub or upside-down bathtub unimodal) failure rates (Xie and Lai [35]). Distributions with such failure rates have attracted a considerable attention of researchers in reliability engineering. In software reliability, bathtub shaped failure rate is encountered in firmware, and in embedded software in hardware devices. Firmware plays an important role in functioning of hard drives of large computers, spacecraft and high performance aircraft control systems, advanced weapon systems, safety critical control systems used for monitoring the industrial process in chemical and nuclear plants (Zhang et al. [36]). The upside down bathtub shaped failure rate is used in data of motor bus failures (Mudholkar et al. [25])), for optimal burn-in decisions (Block and Savits[6]), for ageing properties in reliability (Gupta and Gupta[13], Jiang et al.[16]) and the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The Weibull distribution is a widely used statistical model for studying fatigue and endurance life in engineering devices and materials. Many examples can be found among the electronics, materials, and automotive industries. Recent advances in Weibull theory have also created numerous specialized Weibull applications. Modern computing technology has made many of these techniques accessible across the engineering spectrum. Despite its popularity, and wide applicability the traditional 2-parameters and 3-parameters Weibull distribution is unable to capture the entire lifetime phenomenon for instance the data set which has a non-monotonic failure rate function. Recently several generalization of Weibull distribution has been studied. An approach to the construction of flexible parametric models is to embed appropriate competing models into a larger model by adding shape parameter. Some recent generalizations of Weibull distribution including the exponentiated Weibull, extended Weibull, modified Weibull are discussed in Pham et al. [27] and references therein, along with their reliability functions. The hazard function of the Weibull distribution can only be increasing, decreasing or constant. Thus, it cannot be used to model lifetime data with a bathtub shaped hazard function, such as human mortality and machine life cycles. For many years, researchers have been developing various extensions and modified forms of the Weibull distribution, with different number of parameters. A state of the art survey on the class of such distributions can be found in Laiet al [19]. Xie and Lai [35] proposed a 4-parameters additive Weibull (AW) distribution as a competitive model. A random variable X is said to have an AW distribution if its cumulative distribution function (cdf) is

$$F(x) = 1 - e^{-(\theta x^{\nu} + \gamma x^{\beta}),}\ x \geq 0 \tag{1}$$

where $\beta > 0$ and $\nu > 0$ are shape parameters, and $\theta > 0$ and $\gamma > 0$ are scale parameters.

Elbatal and Aryal [10] introduced the transmuted additive Weibull (TAW) distribution with cumulative distribution function (cdf) and probability density function (pdf) (for$x > 0$) given by

$$F(x) = (1+\lambda)\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right] - \lambda\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^2, \tag{2}$$

and

$$f(x) = \left(\theta \nu x^{\nu-1} + \gamma\beta x^{\beta-1}\right) e^{-(\theta x^{\nu} + \gamma x^{\beta})}\left[1 + \lambda - 2\lambda e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right], \tag{3}$$

where $\beta > 0$ and $\nu > 0$ are shape parameters, and $\theta > 0$ and $\gamma > 0$ are scale parametersand $|\lambda| \leq 1$ is a transmuted parameter.The TAW model shows flexible properties as it contains a lot of well-known distributions as special cases such as exponentiated Weibull, transmuted Weibull, Weibull and linear failure rate distributions.

Many distributions have been made using cumulative distribution function (cdf)$G(x)$, probability density function (pdf)$g(x)$, or survival function $\bar{G}(x)$ that one can rely on, as a baseline distribution, to introduce new models. The Exponentiated generalization is the first generalization allowing for no monotone hazard rates, including the bathtub shaped hazard rate. The cdf of the new distribution is defined by $F(x) = G^{\propto}(x)$,where $\propto > 0$. The exponentiated exponential distribution has been introduced by Ahuja and Nash [2] and further studied by Gupta and Kundu [14]. The first generalization allowing for no monotone hazard rates, including the bathtub shaped hazard rate, is the exponentiated Weibull (EW) distribution due to Mudholkar and Srivastava [24], and Mudholkar et al. [25].

An interesting idea of generalizing a distribution, known in the literature by transmutation, is derived by using the Quadratic Rank Transmutation Map (QRTM) introduced by Shaw and Buckley [30].Merovci [21], introduced transmuted exponentiated exponential distribution.

According to the transmutation generalization approach, the cdf satisfies the relationship

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2. \tag{4}$$

Where G(x) the cdf of the baseline distribution.

This article presents a modification of the transmutation generalization approach given in (4). The proposed modification generalizes the rank of the transmutation map by replacing the constant power by additional parameters. The following definition gives the mechanism of generating a new family of lifetime distributions building on a base model, that is, according to this modification.

Definition 1.1 Let $G(x)$ be the cumulative distribution function (cdf) of a non-negative absolutely continuous random variable, $G(x)$be strictly increasing on its support, and $G(0) = 0$ define a new cdf, F(x), out of $G(x)$as

$$F(x) = (1+\lambda)[G(x)]^{\delta} - \lambda[G(x)]^{\alpha}, x > 0 \tag{5}$$

where $\alpha, \delta > 0$for$0 > \lambda > -1$,and $\alpha > 0$, $(\alpha + \alpha/4) \geq \delta \geq \left(\frac{\alpha}{2}\right)for0 < \lambda < 1$.

This modification due to its flexibility in accommodating all forms of the hazard rate function as seen from Figure (4) (by changing its parameter values) seems to be an important distribution that can be used. Another importance of the proposed model that it is very flexible model that approaches to different distributions when its parameters are changed.

We present special cases of the new family of lifetime distribution.

Exponentiation. for $\lambda = 0$, the distribution function (5) becomes

$$F(x) = [G(x)]^{\delta}, \tag{6}$$

which is the distribution function of the exponentiation**.**

Transmutation. for $\delta = 1 \; and \; \alpha = 2$, the distribution function (5) becomes

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \tag{7}$$

which is the distribution function of the transmutation**.**

Transmutation exponentiation. for $\delta = \alpha/2$, the distribution function (5) becomes

$$F(x) = (1+\lambda)[G(x)]^{\frac{\alpha}{2}} - \lambda[G(x)]^{\propto}, \tag{8}$$

which is the distribution function of the transmutation exponentiation**.**

The rest of the article is organized as follows. In Section2, introduces the proposed a new generalization of the transmuted additive Weibull according to the new class of distribution. In Section 3, we find the reliability function, hazard rate and cumulative hazard rate of the subject model. The Expansion for the pdf and the cdf Functions is derived in Section 4. In section 5, The statistical properties include quantile functions, median , moments and moment generating function are given,. In Section 6, order statistics are discussed. In Section 7, we introduce the method of likelihood estimation as point estimation, give the equation used to estimate the parameters, using the maximum product spacing estimates and the least square estimates techniques. Finally, we fit the distribution to real data set to examine it and to suitability it with nested models.

A New Transmuted Additive Weibull Distribution

In this section, we introduce a new distribution called the new transmuted Additive Weibull distribution denoted by (NTAW) distribution as a generalization of the TAW distribution. The cumulative distribution function of (NTAW) model (for $x > 0$) denoted by $F(x, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) \equiv F(x)$ becomes

$$F(x) = (1+\lambda)\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^{\delta} - \lambda\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^{\alpha}, \tag{9}$$

where as its pdf can be expressed,

$$f(x) = \left(\theta \nu x^{\nu-1} + \gamma\beta x^{\beta-1}\right)e^{-(\theta x^{\nu} + \gamma x^{\beta})}\left[(1+\lambda)\delta\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^{\delta-1} - \lambda\alpha\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^{\alpha-1}\right], \tag{10}$$

where $\beta > 0, \nu, \delta > 0$ and $\alpha > 0$ are shape parameters, and $\theta > 0$ and $\gamma > 0$ are scale parametersand $|\lambda| \leq 1$ is a transmuted parameter. The random variable x with the density function (10) is said to have a new transmuted additive Weibull distribution (NTAW) distribution.

The proposed NTAW model that it is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the NTAW is explained in Table 1 when their parameters are carefully chosen.

Table 1: The special cases of the NTAW distribution

Distribution	**Parameters**							**Author**
	λ	θ	ν	γ	β	δ	α	
TEMW			**1**			$\alpha/2$		Ashour and Eltehiwy [4]
TEAW						$\alpha/2$		
TAW						**1**	**2**	Elbatal and Aryal[10]
EAW	**0**							
AW	**0**					**1**		Xie and Lai [35]
EW	**0**	**0**						Mudholkar and Srivastava[24]

EE	**0**		**1**	**0**	**-**		**-**	Gupta and Kundu[14]
EMW	**0**		**1**				**-**	Elbatal[9]
NTMW					**1**			New
NTW				**0**				New
NTR		**0**			**2**			New
NTLFR			**1**		**2**			New
NTE			**1**	**0**				New
TELFR			**1**		**2**	$\boldsymbol{\alpha/2}$		
TEW		**0**				$\boldsymbol{\alpha/2}$		
TER		**0**			**2**	$\boldsymbol{\alpha/2}$		Merovci[22]
TEE			**1**	**0**	**-**	$\boldsymbol{\alpha/2}$		Merovci[21]
TMW			**1**			**1**	**2**	Khan and King[17]
TLFR			**1**		**2**	**1**	**2**	
TW		**0**				**1**	**2**	Aryal and Tsokos[3]
TR		**0**			**2**	**1**	**2**	Kundu and Raqab[18]
TE			**1**	**0**	**-**	**1**	**2**	Shaw and Buckley[30]
ELFR			**1**	**0**	**-**	**1**	**2**	Sarhan and Kundu[27]
ER	**0**	**0**			**2**		**-**	
MW	**0**		**1**			**1**	**-**	Sarhan and Zaindin[29]
LFR	**0**		**1**		**2**	**1**	**-**	
W	**0**	**0**				**1**	**-**	Weibull[34]
R	**0**	**0**			**2**	**1**	**-**	
E	**0**		**1**	**0**	**-**	**1**	**-**	

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the NTAW distribution for selected values of the parameters $\lambda, \theta, \nu, \gamma, \beta, \delta \; and \; \alpha$ respectively

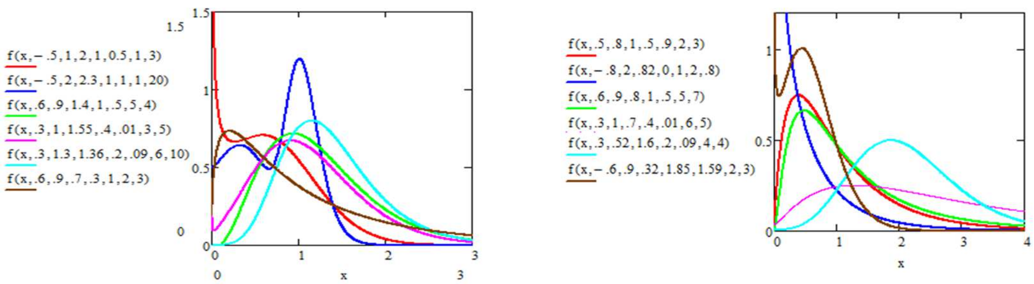


Figure 1: Probability Density Function of the NTAW distribution.

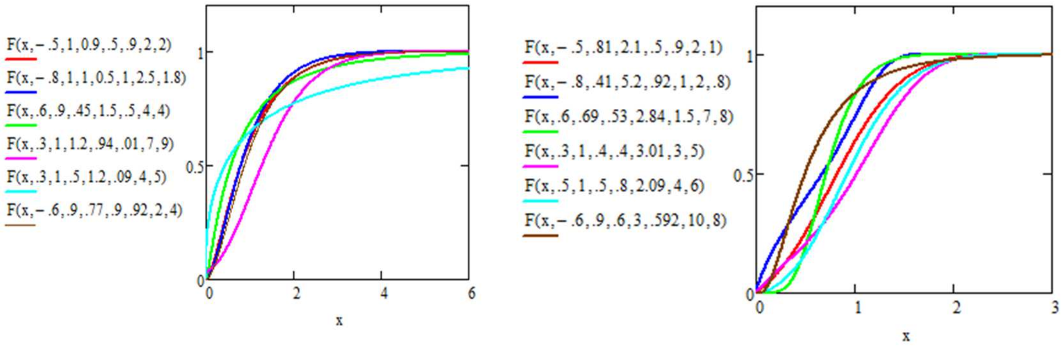


Figure 2: Distribution Function of theNTAW distribution.

Reliability Analysis

The characteristics in reliability analysis which are the reliability function (RF), the hazard rate function (HF) and the cumulative hazard rate function (CHF) for the NTAWD are introduces in this section.

Reliability Function

The reliability function (RF) also known as the survival function, which is the probability of an item not failing prior to some time t, is defined by$\mathrm{R(x)} = 1 - \mathrm{F(x)}$.The reliability function of the NTAW distribution denoted by $\mathrm{R_{NTAW}}(\lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)$, can be a useful characterization of lifetime data analysis. It can be defined as,

$$\mathrm{R_{NTAW}}(x, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) = 1 - \mathrm{F_{NTAW}}(x, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha),$$

the survival function of is given by,

$$\mathrm{R_{NTAW}}(x, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) = 1 - \left[(1+\lambda)\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^{\delta} - \lambda\left[1 - e^{-(\theta x^{\nu} + \gamma x^{\beta})}\right]^{\alpha}\right]. \qquad (11)$$

Figure 3 illustrates the pattern of the called the new transmuted additive Weibull distribution (NTAW) distribution reliability function with different choices of parameters $\lambda, \theta, \nu, \gamma, \beta, \delta$ *and* α .

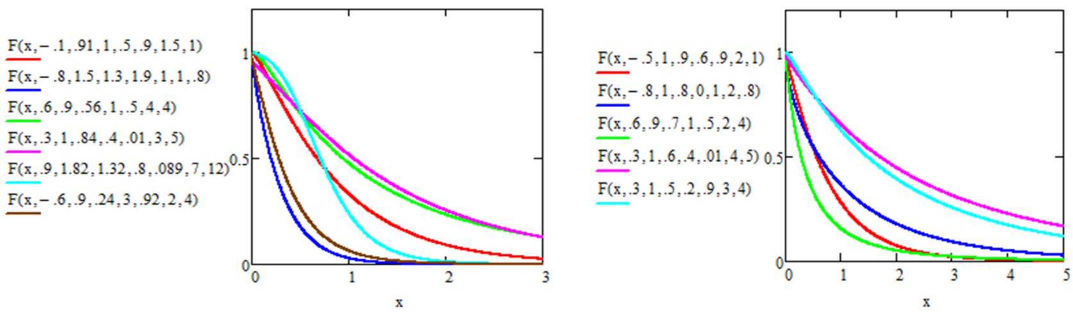


Figure 3: Reliability Function of theNTAW distribution.

Hazard Rate Function

The other characteristic of interest of a random variable is the hazard rate function (HF). the new transmuted additive Weibull distribution also known as instantaneous failure rate denoted by $\mathrm{h_{NTAW}(x)}$, is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the timet. The HF of the NTAWD is defined by

$$\mathrm{h_{NTAW}}(x,\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)=\mathrm{f_{NTAW}}(x,\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)/\mathrm{R_{NTAW}}(x,\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha),$$

$$\mathrm{h_{NTAW}}(x,\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)=\frac{(\theta\nu x^{\nu-1}+\gamma\beta x^{\beta-1})e^{-(\theta x^{\nu}+\gamma x^{\beta})}\left[(1+\lambda)\delta\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha-1}\right]}{(1+\lambda)\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta}-\lambda\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha}}. \quad (12)$$

Figure 4 illustrates some of the possible shapes of the hazard rate function of the new transmuted additive Weibull distribution for different values of the parameters$\lambda,\theta,\nu,\gamma,\beta,\delta$ *and* α.

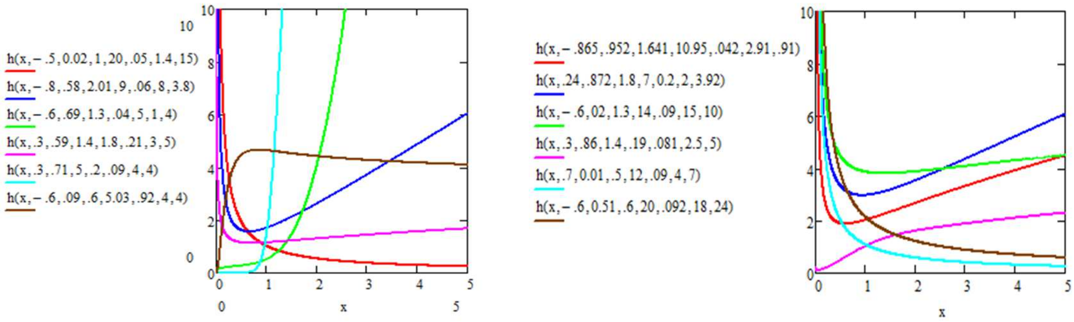


Figure 4: Hazard Rate of the NTAW distribution**.**

Cumulative Hazard Rate Function

The Cumulative hazard function (CHF) of the new transmuted additive Weibull distribution, denoted by $\mathrm{H_{NTAW}}(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)$, is defined as

$$\mathrm{H_{NTAW}}(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)=\int_0^{\mathrm{x}}\mathrm{h_{NTAW}}(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)\mathrm{dx}=-\ln\mathrm{R_{NTAW}}(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha),$$

$$\mathrm{H_{NTAW}}(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)=-\ln\left[1-\left[(1+\lambda)\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta}-\lambda\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha}\right]\right]. \quad (13)$$

Expansion for the pdf and the cdf Functions

In this section, we introduced another expression for the pdf and the cdf functions using. The Maclaurin expansion to simplifying the pdf and the cdf forms.

Expansion for the pdf Function

From equation (10) and using the expansion

$$(1-\mathrm{z})^{\mathrm{k}}=\sum_{\mathrm{j}=0}^{\infty}\frac{(-1)^{\mathrm{j}}\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(\mathrm{k}-\mathrm{j}+1)\mathrm{j}!}\mathrm{z}^{\mathrm{j}}. \quad (14)$$

Which holds for $|\mathrm{z}|<1$ and $\mathrm{k}>0$. Using (14) in Equation. (10), then the pdf function of the new transmuted additive Weibull distribution can be written as:

$$f(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)=\left[(1+\lambda)\delta\sum_{\mathrm{i}=0}^{\infty}\frac{(-1)^{\mathrm{i}}\Gamma(\delta)}{\mathrm{i}!\,\Gamma(\delta-\mathrm{i})}(\theta\nu\mathrm{x}^{\nu-1}+\gamma\beta x^{\beta-1})e^{-(\theta x^{\nu}+\gamma x^{\beta})(\mathrm{i}+1)}\right]$$

$$-\lambda\alpha\sum_{\mathrm{j}=0}^{\infty}\frac{(-1)^{\mathrm{j}}\Gamma(\alpha)}{\mathrm{j}!\,\Gamma(\alpha-\mathrm{j})}(\theta\nu\mathrm{x}^{\nu-1}+\gamma\beta x^{\beta-1})e^{-(\theta x^{\nu}+\gamma x^{\beta})(\mathrm{j}+1)} \quad (15)$$

Expansion for the cdf Function

Using expansion (14) to Equation (9), then the cdf function of the new transmuted additive Weibull distribution can be written as:

$$F(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha) = \left[(1+\lambda)\sum_{\mathrm{i}=0}^{\infty}\frac{(-1)^{\mathrm{i}}\Gamma(\delta+1)}{\mathrm{i}!\,\Gamma(\delta-\mathrm{i}+1)}e^{-(\theta x^{\nu}+\gamma x^{\beta})(\mathrm{i}+1)}\right] - \left[\lambda\sum_{\mathrm{j}=0}^{\infty}\frac{(-1)^{\mathrm{j}}\Gamma(\alpha+1)}{\mathrm{j}!\,\Gamma(\alpha-\mathrm{j}+1)}e^{-(\theta x^{\nu}+\gamma x^{\beta})(\mathrm{j}+1)}\right] \qquad (16)$$

Equation (16) can be written as:

$$F(\mathrm{x},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha) = \left[(1+\lambda)\sum_{\mathrm{i}=0}^{\infty}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{i+k}}\Gamma(\delta+1)}{\mathrm{k}!\,\mathrm{i}!\,\Gamma(\delta-\mathrm{i}+1)}\left((\theta x^{\nu}+\gamma x^{\beta})(\mathrm{i}+1)\right)^{\mathrm{k}}\right] - \left[\lambda\sum_{\mathrm{j}=0}^{\infty}\sum_{\mathrm{m}=0}^{\infty}\frac{(-1)^{\mathrm{j+m}}\Gamma(\alpha+1)}{\mathrm{j}!\,\mathrm{m}!\,\Gamma(\alpha-\mathrm{j}+1)}\left((\theta x^{\nu}+\gamma x^{\beta})(\mathrm{j}+1)\right)^{\mathrm{m}}\right] \qquad (17)$$

Statistical properties

In this section, we discuss the most important statistical properties of the NTAW distribution.

Quantile function

The quantile function is obtained by inverting the cumulative distribution (17), where the p-th quantile x_p of the NTAW model is the real solution of the following equation:

$$(1+\lambda)\sum_{\mathrm{i}=0}^{\infty}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{i+k}}\Gamma(\delta+1)}{\mathrm{k}!\,\mathrm{i}!\,\Gamma(\delta-\mathrm{i}+1)}\left((\theta \mathrm{x_p}^{\nu}+\gamma \mathrm{x_p}^{\beta})(\mathrm{i}+1)\right)^{\mathrm{k}} - \lambda\sum_{\mathrm{j}=0}^{\infty}\sum_{\mathrm{m}=0}^{\infty}\frac{(-1)^{\mathrm{j+m}}\Gamma(\alpha+1)}{\mathrm{j}!\,\mathrm{m}!\,\Gamma(\alpha-\mathrm{j}+1)}\left((\theta \mathrm{x_p}^{\nu}+\gamma \mathrm{x_p}^{\beta})(\mathrm{j}+1)\right)^{\mathrm{m}} - p = 0. \qquad (18)$$

An expansion for the median M follows by taking $p = 0.5$.

Moments

The r^{th}non-central moments $\mu'_{\mathrm{r}} = \mathrm{E}(X^{\mathrm{r}})$or (moments about the origin) are given by theorem 5.1 below:

Theorem 5.1 If X is from a NTAW distribution, then the r^{th}non-central moments is given by

$$\mu'_{\mathrm{r}} = (1+\lambda)\delta\sum_{\mathrm{i}=0}^{\infty}\sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{i+k}}\Gamma(\delta)(\gamma(\mathrm{i}+1))^{\mathrm{k}}}{\mathrm{i}!\,\mathrm{k}!\,\Gamma(\delta-\mathrm{i})}\left[\frac{\theta\nu\Gamma(\frac{\mathrm{r+k\beta+\nu}}{\nu})}{(\theta(\mathrm{i}+1))^{\frac{\mathrm{r+k\beta+\nu}}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{\mathrm{r+k\beta+\beta}}{\nu})}{(\theta(\mathrm{i}+1))^{\frac{\mathrm{r+k\beta+\beta}}{\nu}}}\right] - \lambda\alpha\sum_{\mathrm{j}=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{\mathrm{j}+l}\Gamma(\alpha)(\gamma(\mathrm{j}+1))^{l}}{\mathrm{j}!\,l!\,\Gamma(\alpha-\mathrm{j})}\left[\frac{\theta\nu\Gamma(\frac{\mathrm{r}+l\beta+\nu}{\nu})}{(\theta(\mathrm{j}+1))^{\frac{\mathrm{r}+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{\mathrm{r}+l\beta+\beta}{\nu})}{(\theta(\mathrm{j}+1))^{\frac{\mathrm{r}+l\beta+\beta}{\nu}}}\right].$$

Where $\Gamma(.)$ denote the gamma function, i.e,

$$\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt.$$

Proof:

$$\mu_r'=\mathrm{E}(X^r)=\int_0^\infty X^r f(x,\theta,\gamma,\beta,\delta,\alpha)dx,$$

$$\mu_r'=\int_0^\infty\left\{\left[(1+\lambda)\delta\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\delta)}{i!\,\Gamma(\delta-i)}\left(\theta\nu x^{r+\nu-1}+\gamma\beta x^{r+\beta-1}\right)e^{-(\theta x^\nu+\gamma x^\beta)(i+1)}\right]\right.$$
$$\left.-\lambda\alpha\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\alpha)}{j!\,\Gamma(\alpha-j)}\left[\left(\theta\nu x^{r+\nu-1}+\gamma\beta x^{r+\beta-1}\right)e^{-(\theta x^\nu+\gamma x^\beta)(j+1)}\right]\right\}dx$$

$$\mu_r'=(1+\lambda)\delta\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\delta)}{i!\,\Gamma(\delta-i)}I_1-\lambda\alpha\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\alpha)}{j!\,\Gamma(\alpha-j)}I_2. \tag{19}$$

Now, using

$$e^{-(\gamma x^\beta)(j+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k(i+1)^k\gamma^k x^{k\beta}}{k!}\qquad e^{-(\theta x^\nu)(j+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^k\theta^k x^{k\nu}}{k!}.$$

We have

$$I_1=\int_0^\infty\left(\theta\nu x^{r+\nu-1}+\gamma\beta x^{r+\beta-1}\right)e^{-(\theta x^\nu+\gamma x^\beta)(i+1)}dx$$

$$=\int_0^\infty(\theta\nu x^{r+\nu-1})e^{-(\theta x^\nu)(i+1)}e^{-(\gamma x^\beta)(i+1)}dx+\int_0^\infty\left(\gamma\beta x^{\beta+r-1}\right)e^{-(\theta x^\nu)(i+1)}e^{-(\gamma x^\beta)(i+1)}dx$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(i+1)^k\gamma^k}{k!}\frac{\nu\theta\Gamma(\frac{r+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\nu}{\nu}}}+\sum_{k=0}^{\infty}\frac{(-1)^k(i+1)^k\gamma^{k+1}}{k!}\frac{\beta\Gamma(\frac{r+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\beta}{\nu}}}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(i+1)^k\gamma^k}{k!}\left[\frac{\theta\nu\Gamma(\frac{r+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{r+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\beta}{\nu}}}\right] \tag{20}$$

Similarly , for I_2 we get

$$I_2=\sum_{l=0}^{\infty}\frac{(-1)^l(j+1)^l\gamma^l}{l!}\left[\frac{\theta\nu\Gamma(\frac{r+l\beta+\nu}{\nu})}{(\theta(j+1))^{\frac{r+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{r+l\beta+\beta}{\nu})}{(\theta(j+1))^{\frac{r+l\beta+\beta}{\nu}}}\right] \tag{21}$$

Substituting (20) and (21) in (19) we get

$$\mu_r'=(1+\lambda)\delta\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\Gamma(\delta)(\gamma(i+1))^k}{i!\,k!\,\Gamma(\delta-i)}\left[\frac{\theta\nu\Gamma(\frac{r+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{r+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\beta}{\nu}}}\right]$$

$$-\lambda\alpha\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+l}\Gamma(\alpha)(\gamma(j+1))^{l}}{j!\,l!\,\Gamma(\alpha-j)}\left[\frac{\theta\nu\Gamma(\frac{r+l\beta+\nu}{\nu})}{(\theta(j+1))^{\frac{r+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{r+l\beta+\beta}{\nu})}{(\theta(j+1))^{\frac{r+l\beta+\beta}{\nu}}}\right]. \quad (22)$$

This completes the proof.

In particular, when r = 1, Eq. (22) yields the mean of the NTAW distribution, μ, as

$$\mu=(1+\lambda)\delta\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\Gamma(\delta)(\gamma(i+1))^{k}}{i!\,k!\,\Gamma(\delta-i)}\left[\frac{\theta\nu\Gamma(\frac{k\beta+\nu+1}{\nu})}{(\theta(i+1))^{\frac{k\beta+\nu+1}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{k\beta+\beta+1}{\nu})}{(\theta(i+1))^{\frac{k\beta+\beta+1}{\nu}}}\right]$$
$$-\lambda\alpha\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+l}\Gamma(\alpha)(\gamma(j+1))^{l}}{j!\,l!\,\Gamma(\alpha-j)}\left[\frac{\theta\nu\Gamma(\frac{l\beta+\nu+1}{\nu})}{(\theta(j+1))^{\frac{l\beta+\nu+1}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{l\beta+\beta+1}{\nu})}{(\theta(j+1))^{\frac{l\beta+\beta+1}{\nu}}}\right].$$

The n^{th}central moments or (moments about the mean) can be obtained easily from therth non-central moments throw the relation:

$$m_u=E(X-\mu)^n=\sum_{r=0}^{n}(-\mu)^{n-r}E(X^r).$$

Then the n^{th}central moments of the NTAW is given by:

$$m_u=\sum_{r=0}^{n}(-\mu)^{n-r}\,(1+\lambda)\delta\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\Gamma(\delta)(\gamma(i+1))^{k}}{i!\,k!\,\Gamma(\delta-i)}\left[\frac{\theta\nu\Gamma(\frac{r+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{r+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{r+k\beta+\beta}{\nu}}}\right]$$
$$-\lambda\alpha\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+l}\Gamma(\alpha)(\gamma(j+1))^{l}}{j!\,l!\,\Gamma(\alpha-j)}\left[\frac{\theta\nu\Gamma(\frac{r+l\beta+\nu}{\nu})}{(\theta(j+1))^{\frac{r+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{r+l\beta+\beta}{\nu})}{(\theta(j+1))^{\frac{r+l\beta+\beta}{\nu}}}\right].$$

The Moment Generating Function

Theorem 5.2 If X is from a NTAW distribution, then, its mgf is

$$M_x(t)=(1+\lambda)\delta\sum_{i,m,k=0}^{\infty}\frac{(-1)^{i+k}t^{m}(i+1)^{k}\gamma^{k}\Gamma(\delta)}{k!\,m!\,i!\,\Gamma(\delta-i)}\left[\frac{\theta\nu\Gamma(\frac{m+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{m+k\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{m+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{m+k\beta+\beta}{\nu}}}\right]$$
$$-\lambda\alpha\sum_{j,z,l=0}^{\infty}\frac{(-1)^{j+l}t^{z}(j+1)^{l}\gamma^{l}\Gamma(\alpha)}{z!\,l!\,j!\,\Gamma(\alpha-j)}\left[\frac{\theta\nu\Gamma\left(\frac{z+l\beta+\nu}{\nu}\right)}{\left(\theta(j+1)\right)^{\frac{z+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma\left(\frac{z+l\beta+\beta}{\nu}\right)}{\left(\theta(j+1)\right)^{\frac{z+l\beta+\beta}{\nu}}}\right].$$

Proof:

The moment generating function,$M_x(t)$ can be easily obtained from the rth non-central moment through the relation

$$M_x(t)=\int_{0}^{\infty}e^{tx}f(x,\theta,\gamma,\beta,\delta,\alpha)dx,$$

$$M_x(t)=\int_{0}^{\infty}e^{tx}\left\{\left[(1+\lambda)\delta\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\delta)}{i!\,\Gamma(\delta-i)}\quad(\theta\nu x^{\nu-1}+\gamma\beta x^{\beta-1})e^{-(\theta x^{\nu}+\gamma x^{\beta})(i+1)}\right]\right.$$
$$\left.-\lambda\alpha\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(\alpha)}{j!\,\Gamma(\alpha-j)}\left(\theta\nu x^{\nu-1}+\gamma\beta x^{\beta-1}\right)e^{-(\theta x^{\nu}+\gamma x^{\beta})(j+1)}\right\}dx$$

$$M_x(t)=\left[(1+\lambda)\delta\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\delta)}{i!\,\Gamma(\delta-i)}\int_0^{\infty}\left(\theta\nu x^{\nu-1}+\gamma\beta x^{\beta-1}\right)e^{tx}e^{-(\theta x^{\nu}+\gamma x^{\beta})(i+1)}\,dx\right]$$
$$-\lambda\alpha\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\alpha)}{j!\,\Gamma(\alpha-j)}\int_0^{\infty}\left(\theta\nu x^{\nu-1}+\gamma\beta x^{\beta-1}\right)e^{tx}e^{-(\theta x^{\nu}+\gamma x^{\beta})(j+1)}dx$$

$$M_x(t)=(1+\lambda)\delta\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\delta)}{i!\,\Gamma(\delta-i)}I_1-\lambda\alpha\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\alpha)}{j!\,\Gamma(\alpha-j)}I_2. \tag{23}$$

We have

$$I_1=\int_0^{\infty}\left(\theta\nu x^{\nu-1}+\gamma\beta x^{\beta-1}\right)e^{tx}e^{-(\theta x^{\nu}+\gamma x^{\beta})(i+1)}dx$$
$$=\int_0^{\infty}(\theta\nu x^{r+\nu-1})e^{tx}e^{-(\theta x^{\nu})(i+1)}e^{-(\gamma x^{\beta})(i+1)}dx$$
$$+\int_0^{\infty}\left(\gamma\beta x^{\beta+r-1}\right)e^{tx}e^{-(\theta x^{\nu})(i+1)}e^{-(\gamma x^{\beta})(i+1)}dx$$
$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k t^m (i+1)^k\gamma^k}{k!\,m!}\left[\frac{\theta\nu\Gamma(\frac{m+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{m+k\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{m+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{m+k\beta+\beta}{\nu}}}\right] \tag{24}$$

Similarly , for I_2we get

$$I_2=\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l t^z (j+1)^l\gamma^l}{z!\,l!}\left[\frac{\theta\nu\Gamma(\frac{z+l\beta+\nu}{\nu})}{(\theta(j+1))^{\frac{z+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{z+l\beta+\beta}{\nu})}{(\theta(j+1))^{\frac{z+l\beta+\beta}{\nu}}}\right]. \tag{25}$$

Substituting (24) and (25) in (23), Then, the moment generating function of the NTAW distribution is given by,

$$M_x(t)=(1+\lambda)\delta\sum_{i,m,k=0}^{\infty}\frac{(-1)^{i+k}t^m(i+1)^k\gamma^k\Gamma(\delta)}{k!\,m!\,i!\,\Gamma(\delta-i)}\left[\frac{\theta\nu\Gamma(\frac{m+k\beta+\nu}{\nu})}{(\theta(i+1))^{\frac{m+k\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma(\frac{m+k\beta+\beta}{\nu})}{(\theta(i+1))^{\frac{m+k\beta+\beta}{\nu}}}\right]$$
$$-\lambda\alpha\sum_{j,z,l=0}^{\infty}\frac{(-1)^{j+l}t^z(j+1)^l\gamma^l\Gamma(\alpha)}{z!\,l!\,j!\,\Gamma(\alpha-j)}\left[\frac{\theta\nu\Gamma\left(\frac{z+l\beta+\nu}{\nu}\right)}{\left(\theta(j+1)\right)^{\frac{z+l\beta+\nu}{\nu}}}+\frac{\gamma\beta\Gamma\left(\frac{z+l\beta+\beta}{\nu}\right)}{\left(\theta(j+1)\right)^{\frac{z+l\beta+\beta}{\nu}}}\right].$$

This completes the proof.

Order Statistics

The order statistics and their moments have great importance in many statistical problems and they have many applications in reliability analysis and life testing. The order statistics arise in the study of reliability of a system. The order statistics can represent the lifetimes of units or components of a reliability system. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the NTAW$(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)$ with cumulative distribution function(cdf), and the corresponding probability density function(pdf), as in (9) and (10), respectively. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the corresponding order statistics. Then the pdf of $Y_{(r:n)}$, $1\le r\le n$,denoted by $f_{r:n}(y)$,is given by,

$$f_{r:n}(y)=C_{r:n}f_{NTAW}(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)[F_{NTAW}(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)]^{r-1}[R_{NTAW}(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)]^{n-r}.$$

Then

$$f_{r:n}(X) = C_{r:n}\left[\left(\theta \nu x^{\nu-1} + \gamma\beta x^{\beta-1}\right)e^{-(\theta x^{\nu}+\gamma x^{\beta})}\left[(1+\lambda)\delta\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta-1} - \lambda\alpha\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha-1}\right]\right] *$$
$$(1+\lambda)\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta} - \lambda\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha} *$$
$$\left[1-\left\{(1+\lambda)\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta} - \lambda\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha}\right\}\right]^{n-r} \quad (26)$$

Therefore, the pdf of the largest order statistic $X_n$is given by:

$$f_{X_n(x)=}n\left[\left(\theta \nu x^{\nu-1} + \gamma\beta x^{\beta-1}\right)e^{-(\theta x^{\nu}+\gamma x^{\beta})}\left[(1+\lambda)\delta\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta-1} - \lambda\alpha\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha-1}\right]\right]$$
$$*\left[(1+\lambda)\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta} - \lambda\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha}\right]^{n-1} \quad (27)$$

While, the pdf of the smallest order statistic $X_1$is given by:

$$f_{X_1(x)=}n\left[\left(\theta \nu x^{\nu-1} + \gamma\beta x^{\beta-1}\right)e^{-(\theta x^{\nu}+\gamma x^{\beta})}\left[(1+\lambda)\delta\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta-1} - \lambda\alpha\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha-1}\right]\right] *$$
$$\left[1-\left\{(1+\lambda)\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\delta} - \lambda\left[1-e^{-(\theta x^{\nu}+\gamma x^{\beta})}\right]^{\alpha}\right\}\right]^{n-1} \quad (28)$$

Estimation of the Parameters

In this section, we introduce the method of likelihood to estimate the parameters involved, then give the equations used to estimate the parameters using the maximum product spacing estimates and the least square estimates techniques.

Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) for the parameters of the new transmuted additive Weibull distribution NTAW$(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)$is discussed in this section. Consider the random sample $x_1, x_2, \dots, x_n$ of size n from new transmuted exponentiated additive distribution NTAW$(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)$ with probability density function in (11), then the likelihood function can be expressed as follows

$$L(x_1, x_2, \dots, x_n, \lambda,\theta,\nu,\gamma,\beta,\delta,\alpha) = \prod_{i=1}^{n} f_{\mathrm{NTAW}}(x_i, \lambda,\theta,\nu,\gamma,\beta,\delta,\alpha),$$

$$L(x_1, x_2, \dots, x_n, \lambda,\theta,\nu,\gamma,\beta,\delta,\alpha) = \prod_{i=1}^{n}\left(\theta\nu {x_i}^{\nu-1} + \gamma\beta {x_i}^{\beta-1}\right)e^{-(\theta {x_i}^{\nu}+\gamma {x_i}^{\beta})} *$$
$$\left[(1+\lambda)\delta\left[1-e^{-(\theta {x_i}^{\nu}+\gamma {x_i}^{\beta})}\right]^{\delta-1} - \lambda\alpha\left[1-e^{-(\theta {x_i}^{\nu}+\gamma {x_i}^{\beta})}\right]^{\alpha-1}\right]$$

Hence, the log-likelihood function τ= ln L becomes

$$\tau = \sum_{i=1}^{n} ln\left(\theta\nu {x_i}^{\nu-1} + \gamma\beta {x_i}^{\beta-1}\right) - \sum_{i=1}^{n}\left(\theta {x_i}^{\nu} + \gamma {x_i}^{\beta}\right) +$$
$$\sum_{i=1}^{n} ln\left[(1+\lambda)\delta\left[1-e^{-(\theta {x_i}^{\nu}+\gamma {x_i}^{\beta})}\right]^{\delta-1} - \lambda\alpha\left[1-e^{-(\theta {x_i}^{\nu}+\gamma {x_i}^{\beta})}\right]^{\alpha-1}\right] \quad (29)$$

Differentiating Equation (29) with respect to $\lambda,\theta,\nu,\gamma,\beta,\delta$ andαthen equating it to zero, we obtain the MLEs of $\lambda,\theta,\nu,\gamma,\beta,\delta$ andα as follows,

$$\frac{\partial\tau}{\partial\lambda}=\sum_{i=1}^{n}\left[\frac{\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}}{\left[(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}\right]}\right], \tag{30}$$

$$\frac{\partial\tau}{\partial\theta}=\sum_{i=1}^{n}\frac{\nu x_i{}^{\nu-1}}{(\theta\nu x_i{}^{\nu-1}+\gamma\beta x_i{}^{\beta-1})}-\sum_{i=0}^{\infty}x_i{}^{\nu}$$
$$+\sum_{i=1}^{n}\left[\frac{x_i{}^{\nu}e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\left[\lambda(\delta-1)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-2}-\lambda(\alpha-1)\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-2}\right]}{(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}}\right], \tag{31}$$

$$\frac{\partial\tau}{\partial\nu}=\sum_{i=1}^{n}\frac{x_i{}^{\nu-1}\theta(1+\nu\ln x_i)}{(\theta\nu x_i{}^{\nu-1}+\gamma\beta x_i{}^{\beta-1})}-\sum_{i=0}^{\infty}x_i{}^{\nu}\,\theta\ln x_i$$
$$+\sum_{i=1}^{n}\frac{e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}(x_i{}^{\nu}\,\theta\ln x_i)\left[\lambda(\delta-1)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-2}-\lambda(\alpha-1)\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-2}\right]}{(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}} \tag{32}$$

$$\frac{\partial\tau}{\partial\gamma}=\sum_{i=1}^{n}\frac{\beta x_i{}^{\beta-1}}{(\theta\nu x_i{}^{\nu-1}+\gamma\beta x_i{}^{\beta-1})}-\sum_{i=0}^{\infty}x_i{}^{\beta}+$$
$$\sum_{i=1}^{n}\left[\frac{x_i{}^{\beta}e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\left[\lambda(\delta-1)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-2}-\lambda(\alpha-1)\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-2}\right]}{\left[(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}\right]}\right], \tag{33}$$

$$\frac{\partial\tau}{\partial\beta}=\sum_{i=1}^{n}\frac{x_i{}^{\beta-1}\gamma(1+\beta\,ln\,x_i)}{(\theta\nu x_i{}^{\nu-1}+\gamma\beta x_i{}^{\beta-1})}-\sum_{i=0}^{\infty}x_i{}^{\beta}\,\gamma ln\,x_i$$
$$+\sum_{i=1}^{n}\frac{e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\left(x_i{}^{\beta}\,\gamma ln\,x_i\right)\left[\lambda(\delta-1)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-2}-\lambda(\alpha-1)\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-2}\right]}{(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}} \tag{34}$$

$$\frac{\partial\tau}{\partial\delta}=\sum_{i=1}^{n}\frac{(1+\lambda)\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}\left\{\delta\,ln\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]+1\right\}}{\left[(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}\right]}, \tag{35}$$

and

$$\frac{\partial\tau}{\partial\alpha}=\sum_{i=1}^{n}\frac{(-1)\lambda\left[1-e^{-(\theta x_i+\gamma x_i{}^{\beta})}\right]^{\alpha-1}\left\{\alpha\ln\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]+1\right\}}{\left[(1+\lambda)\delta\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\delta-1}-\lambda\alpha\left[1-e^{-(\theta x_i{}^{\nu}+\gamma x_i{}^{\beta})}\right]^{\alpha-1}\right]}. \tag{36}$$

The maximum likelihood estimator $\underline{\hat{\vartheta}}(\hat{\lambda},\hat{\theta},\hat{\nu},\hat{\gamma},\hat{\beta},\hat{\delta},\hat{\alpha})=$ of $\underline{\vartheta}=(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)$ is obtained by solving the nonlinear system of equations (30) through (36). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin [5]. This method is based on an idea that the differences (Spacing) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i}, \tag{37}$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) dx \, ; \quad i = 1,2, \dots, n+1, \tag{38}$$

where, $F(x_{(0)}, \lambda, , \nu, \theta, \gamma, \beta, \delta, \alpha) = 0$ and $F(x_{(n+1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) = 0$. The MPS estimators $\hat{\lambda}_{PS}$, $\hat{\theta}_{PS}$, $\hat{\nu}_{PS}$, $\hat{\gamma}_{PS}$, $\hat{\beta}_{PS}$, $\hat{\delta}_{PS}$ and $\hat{\alpha}_{PS}$ of $\lambda, \theta, \nu, \gamma, \beta, \delta$ and α are obtained by maximizing the geometric mean (*GM*) of the differences. Substituting pdf of NTAW distribution in (38) and taking logarithm of the above expression, we will have

$$\log GM = \frac{1}{n+1} \sum_{i=1}^{n+1} \log\left[F(x_{(i)}\lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)\right]. \tag{39}$$

The MPS estimators $\hat{\lambda}_{PS}$, $\hat{\theta}_{PS}$, $\hat{\nu}_{PS}$, $\hat{\gamma}_{PS}$, $\hat{\beta}_{PS}$, $\hat{\delta}_{PS}$ and $\hat{\alpha}_{PS}$ of $\lambda, \theta, \nu, \gamma, \beta, \delta$ and αcan be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial \, log \, GM}{\partial \lambda} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\lambda}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\lambda}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

$$\frac{\partial \, log \, GM}{\partial \theta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\theta}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\theta}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

$$\frac{\partial \, log \, GM}{\partial \nu} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\nu}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\nu}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

$$\frac{\partial \, log \, GM}{\partial \gamma} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\gamma}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\gamma}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

$$\frac{\partial \, log \, GM}{\partial \beta} = \frac{1}{n+1} \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\beta}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\beta}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

$$\frac{\partial \, log \, GM}{\partial \delta} = \frac{1}{n+1} \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\delta}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\delta}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

and

$$\frac{\partial \, log \, GM}{\partial \alpha} = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{F'_{\alpha}(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F'_{\alpha}(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)} \right] = 0,$$

Least square estimates

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the ordered sample of size n drawn the NTAW distribution. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+1}; \quad i = 1,2,\dots,n. \tag{40}$$

The least square estimates $\hat{\lambda}_{LS}, \hat{\theta}_{LS}, \hat{\nu}_{LS}, \hat{\gamma}_{LS}, \hat{\beta}_{LS}, \hat{\delta}_{LS}$ and $\hat{\alpha}_{LS}$ of $\lambda, \theta, \nu, \gamma, \beta, \delta$and α are obtained by minimizing

$$Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha) = \sum_{i=1}^{n}\left[F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right]^2.$$

Therefore, $\hat{\lambda}_{LS}, \hat{\theta}_{LS}, \hat{\nu}_{LS}, \hat{\gamma}_{LS}, \hat{\beta}_{LS}, \hat{\delta}_{LS}$ and $\hat{\alpha}_{LS}$ of $\lambda, \theta, \nu, \gamma, \beta, \delta$and α can be obtained as the solution of the following system of equations:

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\lambda} = \sum_{i=1}^{n} F_{\lambda}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\theta} = \sum_{i=1}^{n} F_{\theta}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\nu} = \sum_{i=1}^{n} F_{\nu}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\gamma} = \sum_{i=1}^{n} F_{\gamma}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\beta} = \sum_{i=1}^{n} F_{\beta}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\delta} = \sum_{i=1}^{n} F_{\delta}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

and

$$\frac{\partial Z(\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha)}{\partial\alpha} = \sum_{i=1}^{n} F_{\alpha}^{'}\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\nu,\gamma,\beta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve non-linear optimization problem are available in R [33]. We used nlm( ) package for optimizing (29).

Applications

In this section, we use two real data sets to to see how the new model works in practice.compare the fits of the NTAW distribution with others models. In each case, the parameters are estimated by maximum likelihood as described in Section 7, using the R code.

Data Set 1

The first data set represents the ages for 155 patients of breast tumors taken from (June-November 2014), whose entered in (Breast Tumors Early Detection Unit, Benha Hospital University, Egypt).

Table 2: The ages for 155 patients of breast tumors

46	32	50	46	44	42	69	31	25	29	40	42	24	17	35
48	49	50	60	26	36	56	65	48	66	44	45	30	28	40
40	50	41	39	36	63	40	42	45	31	48	36	18	24	35
30	40	48	50	60	52	47	50	49	38	30	52	52	12	48
50	45	50	50	50	53	55	38	40	42	42	32	40	50	58
48	32	45	42	36	30	28	38	54	90	80	60	45	40	50
50	40	50	50	50	60	39	34	28	18	60	50	20	40	50
38	38	42	50	40	36	38	38	50	50	31	59	40	42	38
40	38	50	50	50	40	65	38	40	38	58	35	60	90	48
58	45	35	38	32	35	38	34	43	40	35	54	60	33	35
36	43	40	45	56										

In order to compare the two distribution models, we consider criteria like $-2\mathcal{L}$, AIC (Akaike information criterion), $\mathrm{AIC_C}$ (corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller$-2\mathcal{L}$, AIC and $\mathrm{AIC_C}$ values:

$$\mathrm{AIC} = -2\mathcal{L} + 2k,$$

$$\mathrm{AIC}_C = -2\mathcal{L} + \left(\frac{2kn}{n-k-1}\right),$$

and

$$\mathrm{BIC} = -2\mathcal{L} + k\log(n),$$

where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.

Table 3 shows the parameter estimation based on the maximum likelihood and gives the values of the criteria AIC, $\mathrm{AIC_C}$, and BIC test. The values in Table 2 indicate that the NTAW distribution leads to a better fit over all the other models.

Table 3. MLEs the measures AIC, $\mathrm{AIC_C}$ and BIC test to 155 patients of breast tumors data for the models

Model	**Parameter Estimates**	$-\boldsymbol{logL}$	**AIC**	$\mathbf{AIC_C}$	**BIC**
NTAW	$\boldsymbol{\lambda = 0.007700}$	**601.8007**	**1217.601**	**1218.363**	**1238.905**
	$\boldsymbol{\theta = 0.0005621}$				
	$\boldsymbol{\nu} = \mathbf{2.1463001}$				
	$\boldsymbol{\gamma = 0.083400}$				
	$\boldsymbol{\beta = 0.011520}$				

	$\delta = 2.199999$ $\alpha = 2.845200$				
TAW	$\lambda = 0.007699$ $\theta = .00027438$ $\nu = 2.146299$ $\gamma = 0.0833996$ $\beta = 0.0115199$	**656.6481**	**1323.296**	**1323.699**	**1338.513**
TEMW	$\lambda - 0.036545$ $\theta = 0.0022057$ $\gamma = 0.0220319$ $\beta = 0.073650$ $\alpha = 0.2476071$	**628.6509**	**1267.302**	**1267.705**	**1282.519**
EMW	$\theta = 0.439622$ $\gamma = 0.52237$ $\beta = 0.92627$ $\alpha = 3.74042$	**613.903**	**1235.806**	**1236.073**	**1247.98**
AW	$\theta = 0.0002475$ $\nu = 2.146300$ $\gamma = 0.08339602$ $\beta = 0.4151989$	**688.4355**	**1384.871**	**1385.138**	**1397.045**
MW	$\theta = 0.00721750$ $\gamma = 0.015028$ $\beta = 1.011300$	**739.2161**	**1484.432**	**1484.591**	**1493.562**
W	$\gamma = 3.6871004$ $\beta = 0.02078562$	**610.2967**	**1224.593**	**1224.672**	**1230.68**

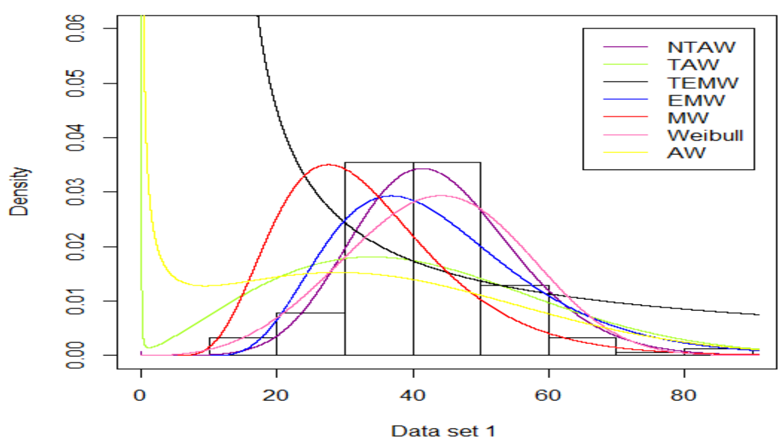


Figure 5: Estimated densities of the data set 1.

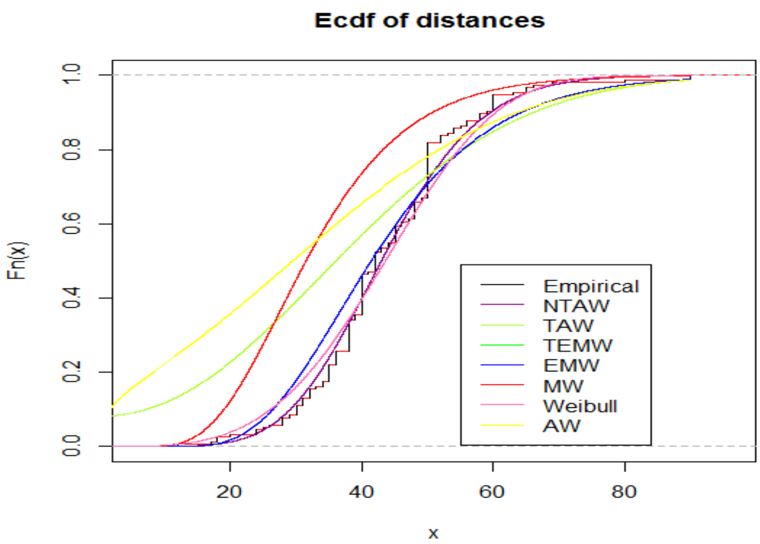
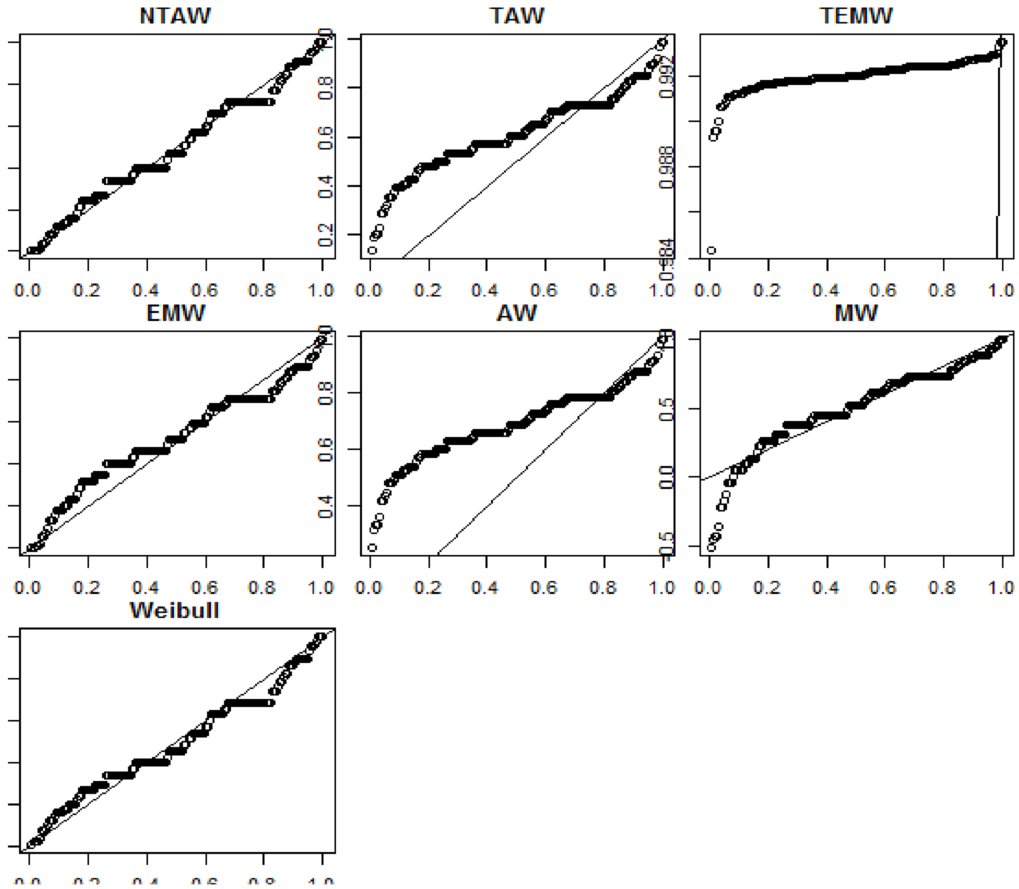


Figure 6: Empirical, fitted NTAW, TAW, TEMW, EMW, MW, Weibull, and AW of the data set 1.



Figuren7: Probability plots for NTAW, TAW, TEMW, EMW, MW, Weibull, and Additive Weibull of the data set 1.

Data Set 2

The second data set represents failure time of 50 items reported in Aarset [1].

Some summary statistics for the failure time data are as follows:

Min.	**1^{st} Qu.**	**Median**	**Mean**	**3^{rd} Qu.**	**Max.**
0.10	13.50	48.50	45.67	81.25	86.00

Table 4. MLEs the measures AIC, AIC_C and BICS test to failure time data for the models

Model	Parameter Estimates	$-logL$	AIC	AIC_C	BIC
NTAW	$\lambda = -0.08220037$	213.138	440.2776	442.9443	453.6618
	$\theta = 1.778 * 10^{-5}$				
	$\nu = 2.150033$				
	$\gamma = 8.922 * 10^{-5}$				
	$\beta = 0.404211$				
	$\delta = 0.317617$				
	$\alpha = 0.00510335$				
TAW	$\lambda = 0.0076999983$	229.3821	468.7642	470.1278	478.3243
	$\theta = 0.00010750$				
	$\nu = 2.1463000$				
	$\gamma = 0.083400002$				
	$\beta = 0.4152000011$				
TEMW	$\lambda = -0.1640672$	236.6535	487.6286	488.992	497.1887
	$\theta = 0.0176781$				
	$\gamma = 0.00193298$				
	$\beta = 0.03926070$				
	$\alpha = 0.949241462$				
EMW	$\theta = 0.018673571$	238.8143	481.307	482.1959	488.9551
	$\gamma = 0.001822666$				
	$\beta = 0.010505798$				
	$\alpha = 0.703411609$				
AW	$\theta = 0.0002399$	237.7583	483.5166	484.4055	491.1647
	$\nu = 1.852800617$				
	$\gamma = 0.016255755$				
	$\beta = 0.9475073247$				
MW	$\theta = 1.827194$	241.0289	488.0578	488.5795	493.7939
	$\gamma = 1.80309$				
	$\beta = 1.000288$				
W	$\gamma = 0.9489561$	240.9796	485.959	486.2145	489.7832
	$\beta = 0.02227559$				

These results indicate that the NTAW model has the lowest AIC and AIC$_C$ and BIC values among the fitted models. The values of these statistics indicate that the NTAW model provides the best fit to this data.

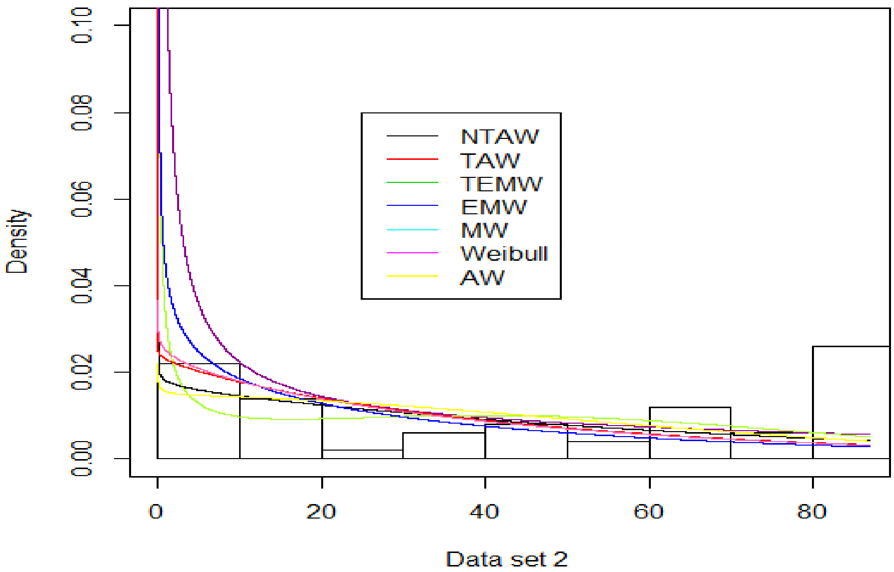


Figure 8: Estimated densities of the data set 2.

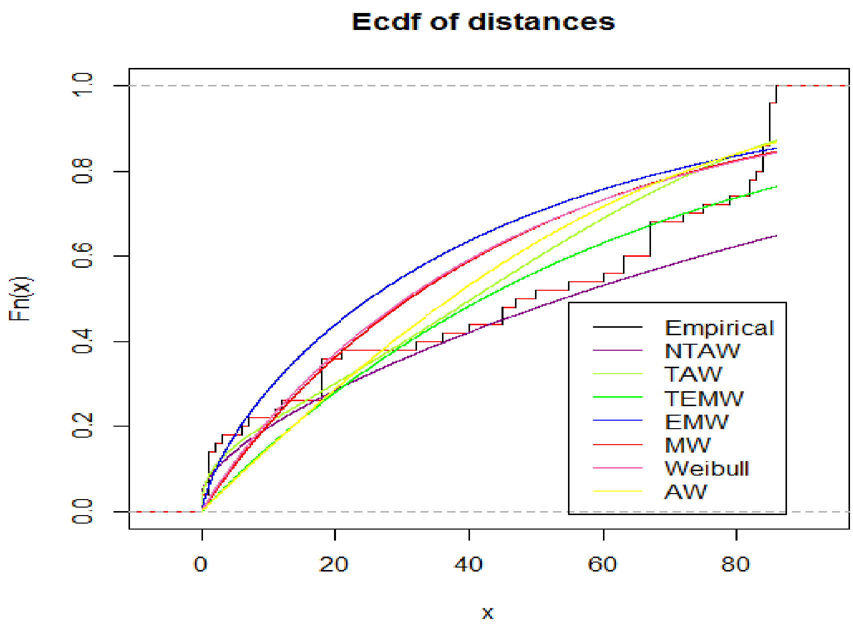


Figure 9: Empirical, fitted NTAW, TAW, TEMW, EMW, MW, Weibull, and AW of the data set 2.

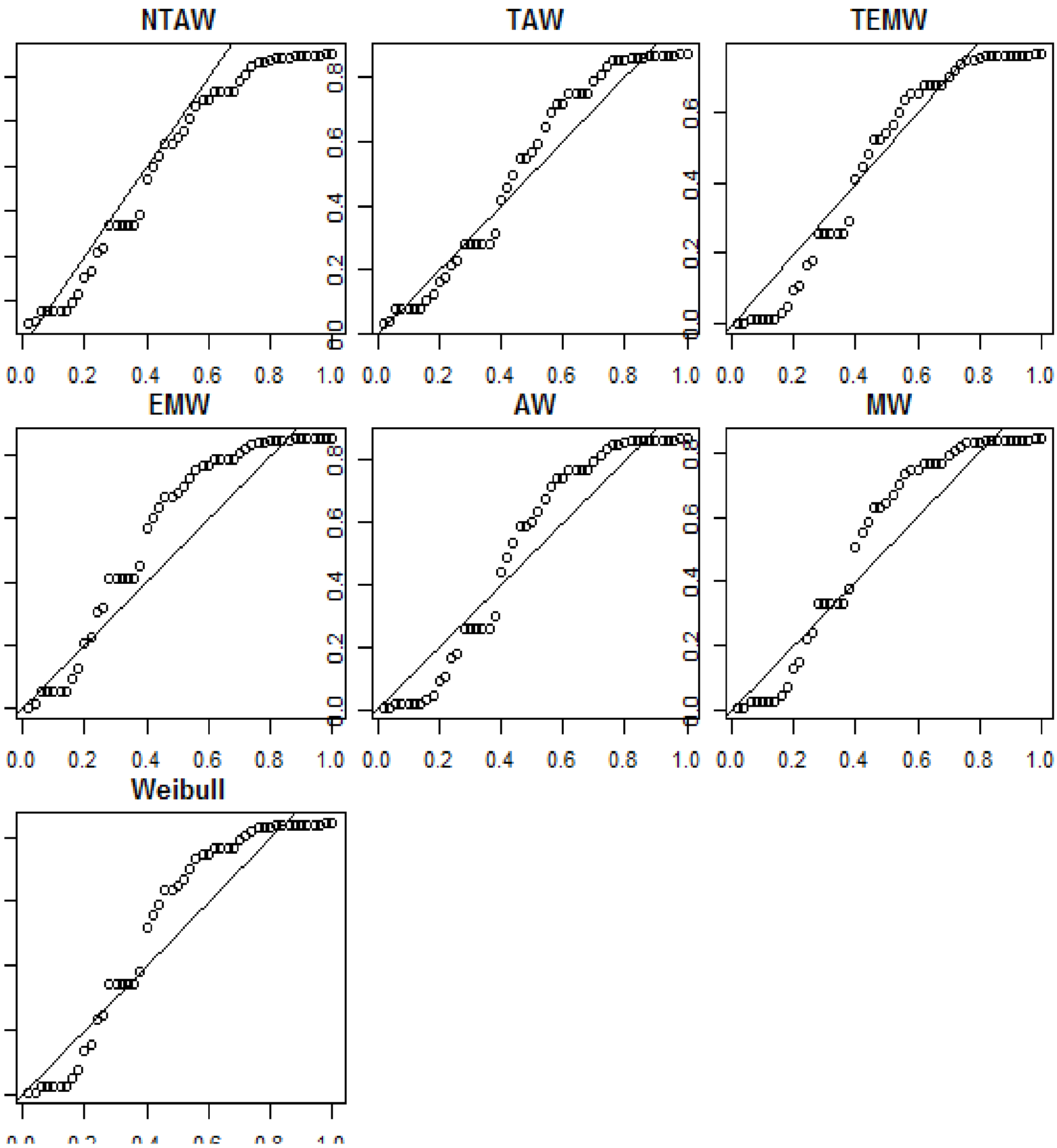


Figure 10: Probability plots for NTAW, TAW, TEMW, EMW, MW, Weibull, and AW of the data set 2.

Conclusions

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modeling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this paper, we introduce a new transmuted additive Weibull distribution obtained using a new family of lifetime distribution as generalization technique. We refer to the new model as the NTAW distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model, explicit expressions for the moments. The model parameters are estimated by maximum likelihood. The new model is compared with some models and provides consistently better fit than other classical lifetime models. We hope that the proposed family will serve as a reference and help to advance future research in this area.

References

[1] Aarset, M. V., 1987,"How to identify a bathtub hazard rate," IEEE Transactions on Reliability, 36(1), pp. 106-108.

[2] Ahuja, J.C. and Nash, S.W., 1967,"The generalized Gompertz-Verhulst family of distributions," Sankhya, 29, pp. 141-161.

[3] Aryal, G. R., & Tsokos, C. P., 2011,"Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution" European Journal of Pure and Applied Mathematics, 4(2), pp. 89-102.

[4] Ashour, S.K. and Eltehiwy, M.A., 2013,"Transmuted Exponentiated Modified Weibull Distribution," International Journal of Basic and Applied Sciences, 2 (3), pp. 258-269.

[5] Barlow, R. E., & Proschan, F., 1981. Statistical Theory or Reliability and Life Testing: Probability Models. Tobe with.

[6] Block, H. W., & Savits, T. H., 1997,"Burn-in. Statistical Science," 12(1), pp. 1-19.

[7] Cheng, R. C. H., &Amin, N. A. K., 1983, "Estimating parameters in continuous univariate distributions with a shifted origin'" Journal of the Royal Statistical Society. Series B (Methodological), pp. 394-403.

[8] Cordeiro, G. M., & de Castro, M., 2011,"A new family of generalized distributions," Journal of statistical computation and simulation, 81(7), pp. 883-898.

[9] Elbatal, I., 2011, "Exponentiated Modified Weibull Distribution," Economic Quality Control, 26(2), pp. 189-200.

[10] Elbatal, I., and Aryal, G., 2013, "On the Transmuted Additive Weibull Distribution," AUSTRIAN J OURNAL OF STATISTICS, 42(2), pp. 117-132.

[11] Greenwich, M., 1992,"A unimodal hazard rate function and its failure distribution," Statistical Papers, 33(1), pp. 187-202.

[12] Gupta, R. C., Gupta, P. L., & Gupta, R. D., 1998,"Modeling failure time data by Lehman alternatives," Communications in Statistics-Theory and methods, 27(4),pp. 887-904.

[13] Gupta, P. L., & Gupta, R. C., 1983,"On the moments of residual life in reliability and some characterization results," Communications in Statistics-Theory and Methods, 12(4), pp. 449-461.

[14] Gupta, R. D., & Kundu, D., 1999,"Generalized exponential distributions," Australian & New Zealand Journal of Statistics, 41(2), pp.173-188.

[15] Gupta, R. D., & Kundu, D., 2001, "Exponentiated exponential family: An alternative to gamma and Weibull distributions," Biometrical journal, 43(1), pp. 117-130.

[16] Jiang, R., Ji, P., & Xiao, X., 2003,"Aging property of unimodal failure rate models," Reliability Engineering & System Safety, 79(1), pp. 113-116.

[17] Khan, M. S., & King, R., 2013, Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution," European Journal of Pure and Applied Mathematics, 6(1), pp. 66-88.

[18] Kundu, D., & Raqab, M. Z., 2005,"Generalized Rayleigh distribution: different methods of estimations," Computational statistics & data analysis, 49(1), pp. 187-200.

[19] Lai, C. D., Xie, M., & Murthy, D. N. P., 2003,"A modified Weibull distribution," Reliability, IEEE Transactions on, 52(1), pp. 33-37.

[20] Marshall, A. W., & Olkin, I., 1997,"A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families," Biometrika, 84(3), pp. 641-652.

[21] Merovci, F., 2013,"Transmuted exponentiated exponential distribution,"Mathematical Sciences and Applications E-Notes, 1(2), pp. 112-122.

[22] Merovci, F., 2014,"Transmuted generalized Rayleigh distribution," Journal of Statistics Applications and Probability, 3(1), pp. 9-20.

[23] Merovci, F. and Elbatal, I., 2014,"Transmuted Lindley-Geometric Distribution and its Applications," J. Stat. Appl. Pro. 3, No. 1, pp. 77-91.

[24] Mudholkar, G. S., & Srivastava, D. K., 1993,"Exponentiated Weibull family for analyzing bathtub failure-rate data," IEEE Transactions on Reliability, 42(2), pp. 299-302.

[25] Mudholkar, G. S., Srivastava, D. K., & Freimer, M., 1995,"The exponentiated Weibull family: a reanalysis of the bus-motor-failure data," Technometrics, 37(4), pp. 436-445.

[26] Murthy, D. P., Xie, M., & Jiang, R., 2004. Weibull models. John Wiley & Sons.

[27] Pham, H., & Lai, C. D., 2007,"On recent generalizations of the Weibull distribution," IEEE Transactions on Reliability, 56(3), pp. 454-458.

[28] Sarhan, A. M., & Kundu, D., 2009," linear failure rate distribution," Communications in Statistics—Theory and Methods, 38(5), pp. 642-660.

[29] Sarhan, A. M., & Zaindin, M., 2009, "Modified Weibull distribution," Applied Sciences, 11(1), pp. 123-136.

[30] Shaw, W. T., Buckley, I. R., 2009, "The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map," arXivpreprint arXiv:0901.0434.

[31] Silva, G. O., Ortega, E. M., & Cordeiro, G. M., 2010,"The beta modified Weibull distribution," Lifetime data analysis, 16(3), pp. 409-430.

[32] Surles, J. G., & Padgett, W. J., 2001,"Inference for reliability and stress-strength for a scaled Burr type X distribution," Lifetime Data Analysis, 7(2), pp. 187-200.

[33] Team, R. C., 2012. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, 2012. ISBN 3-900051-07-0.
[34] Weibull, W., 1939,"A statistical theory of the strength of materials," IVA Handlingar (Royal Swedish Academy of Engineering, Proceedings), 151.
[35] Xie, M., & Lai, C. D., 2006. Stochastic ageing and dependence for reliability. Springer, New York.
[36] Zhang, T., Xie, M., Tang, L. C., & Ng, S. H., 2005,"Reliability and modeling of systems integrated with firmware and hardware," International Journal of Reliability, Quality and Safety Engineering, 12(03), pp. 227-239.