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STATISTICALLY CONVERGENT MULTIPLE SEQUENCES IN PROBABILISTIC NORMED SPACES

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In this paper we define concepts of statistically convergent and statistically Cauchy multiple sequences in probabilistic normed spaces. We prove a useful characterization for statistically convergent multiple sequences. We will introduce the statistical limit points, statistical cluster points in probabilistic normed spaces. Moreover we will give the relation between them and limit points of multiple sequences in probabilistic normed spaces.

Keywords: Density; Statistical convergence; Continuous t-norm; Probabilistic normed spaces.

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1. Introduction

The notion of sequence spaces was extended to double sequences in the beginning of nineteenth century by Pringsheim [14]. Initial works on double sequence is found in Browmich [2]. Hardy [6] introduced the notion of regular convergence for double sequences. Moricz[12] studied some properties of double sequences of real and complex numbers. Recently different types of double sequences have been introduced and investigated from different aspects by Basarir, and Sonalcan [1], Moricz and Rhoades [13], Tripathy [19], Tripathy and Sarma [25, 27] and many others.

In the recent past sequence spaces have been investigated from different aspects. From fuzzy set theory point of view by Tripathy and Baruah [20, 21], Tripathy and Borgohain [22, 23], Tripathy and Dutta [24], Tripathy and Sarma [26], Tripathy, Sen and Nath [28] and many others.

Metric spaces are sets in which there is defined a notion of distance between pair of points. The concept of an abstract metric space was formulated in 1906 by Frechet [5], which furnishes a common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of distance appears. The object under consideration may be points, functions, sets, and the subjective experiences of sensations. There is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set and numbers associated with each pair of elements satisfying certain conditions. But in reality, the instances in which the theory of metric spaces has been applied is an over idealization. Therefore in such situations it is appropriate to look upon the distance concept as a statistical rather than a deterministic one. More precisely, instead of associating a number calledthe distance $d(p, q)$-with every pair of elementp, q, one

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should associate a distribution function F_{pq} and for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x. This generalizes the concept of a metric space. This generalization which was introduced by Menger [9] and named as statistical metric space.

Menger [9] gave postulates for the distribution functions F_{pq}. These include a generalized triangle inequality. In addition, he constructed a theory of betweeness and indicated possible fields of application. In 1943, after Wald [29] improved Mengers notion and introduced the notion of generalized triangle inequality and proposed an alternative definition. On the basis of this new inequality, Wald [29] constructed a theory of betweeness having certain advantages over Mengers theory. Later on Menger [10], considered Walds version of triangle inequality for his investigations in Probalistic normed space. For some detailed account one may refer to Constantin and Istratescu [3], Menger [11] and Sklar [17, 18]. Statistical convergence of single and double sequences in probabilistic normed spaces has been introduced and studied by Karakus [7, 8]. In this paper we extend this notion to multiple sequences.

2. Statistical Convergence of Multiple Sequences in Probabilistic Normed Spaces

Definition 1. A function $f : \mathbb{R}^+ \to \mathbb{R}_0^+$ is called a distribution function if it is a non-decreasing, left continuous on its domain with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.
Throughout D denotes the set of all distribution functions.

Definition 2. A triangular norm or a t-norm is a binary operation on $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$
1)$a * 1 = a$,
2)$a * b = b * a$,
3)$c * d \geq a * b$ if $c \geq a$ and $d * b$,
4)$(a * b) * c = a * (b * c)$.

Example 1. Consider the $*$ operation $a*b = \max a + b - 1, 0$. Then $*$ is a t-norm. Similarly one can consider $a * b = ab$, $a * b = \min\{a, b\}$ on $[0, 1]$ and verify that these are also t-norms.

Definition 3. A triplet $(X, N, *)$ is called a probabilistic normed space (in short PN-space), if X is a real vector space, $N : X \to D$ (for $x \in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t \in \mathbb{R}$) and $*$, a t-norm satisfying the following conditions:
(i) $N_x(0) = 0$,
(ii) $N_x(t) = 1$, for all $t > 0$ if and only if $x = 0$,
$(iii) N_{\alpha x}(t) = N_x\left(\frac{t}{|\alpha|}\right)$, for all $\alpha \in \mathbb{R} - \{0\}$,
(iv) $N_{x+y}(s + t) \geq N_x(s) * N_y(t)$, for all $x, y \in X$ and $s, t \in \mathbb{R}^+$.

Example 2. Let $(X, \|.\|)$ be a normed linear space and $\mu \in D$ with $\mu(0) = 0$ and $\mu \neq h$ where
$h(t) = \begin{cases} 0, for\ all\ t \leq 0; \\ 1,\ for\ all\ t > 0 \end{cases}$. Define $N_x(t) = \begin{cases} h(t),\ if\ x = 0 \\ \mu(\frac{t}{\|x\|})\ if\ x \neq 0 \end{cases}$ where $x \in X$ and $t \in \mathbb{R}$.
Then $(X, N, *)$ is a PN space.

We define a function μ on $\mathbb{R}$ by $\mu(x) = \begin{cases} 0,\ x \leq 0 \\ \frac{x}{1+x},\ x > 0 \end{cases}$.

Then we obtain the following PN
$$N_x(t) = \begin{cases} h(t),\ x = 0 \\ \frac{t}{t+\|x\|},\ x \neq 0 \end{cases}$$

Definition 4. A multiple sequence $x = (x_{n_1n_2....n_k})$ is said to be convergent to $L \in X$ with respect to N if for every $\varepsilon > 0$ and $\beta \in (0,1)$, there exists a positive integer m_0 such that

$$N_{x_{n_1n_2....n_k}-L}(\varepsilon) > 1-\beta, \text{ whenever } n_i \geq m_0, \text{ for all } i = 1,2,3,...,k.$$

It is denoted by $N - \lim x_{n_1n_2....n_k} = L$.

Definition 5. A multiple sequence $x = (x_{n_1n_2....n_k})$, is said to be a Cauchy sequence with respect to N if for every $\varepsilon > 0$ and $\beta \in (0,1)$, there exists a positive integer m_0 such that

$$N_{x_{n_1n_2....n_k}-x_{l_1l_2....l_k}}(\varepsilon) > 1-\beta, \text{ whenever } n_i \geq m_0,\ l_i \geq m_0 \text{ for all } i = 1,2,3,..., k.$$

The notion of statistical convergence was studied by Fast[4] and Schoenberg [16] independently in 1950s. Later on it was studied by Salat [15]. The notion of statistically convergent double sequences was introduced by Tripathy [7]. In this article we introduce the notion of asymptotic density for subsets of $\mathbb{N}^k$.

Definition 6. A subset $E \subset \mathbb{N}^k$ is said to have asymptotic density $\delta_k(E)$ if

$$\lim_{n_1,n_2,....,n_k} \frac{1}{n_1n_2...n_k} \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ...... \sum_{i_k=1}^{n_k} \chi_E(i_1,i_2,....,i_k) \text{ exists.}$$

For example if we consider the set
$K = \{(n_1,n_2,...,n_k) \in \mathbb{N}^k : n_1,n_2,...,n_k = i^2,\ i \in \mathbb{N}\}$ then,
$\delta_k(K) = \lim\limits_{n_1,n_2,...,n_k} \frac{\sqrt{n_1}\sqrt{n_2}...\sqrt{n_k}}{n_1n_2...n_k} = 0.$

Note: For $i = 1$, it is the usual asymptotic density of subsets of $\mathbb{N}$. For $i = 2$, it is the double asymptotic density of subsets of $\mathbb{N}\times\mathbb{N}$. For $i = 3$, it is the triple asymptotic density.

Definition 7. A subset $K \subset \mathbb{N}^k$ is said to have upper asymptotic density $\bar{\delta}_k(K)$ if

$$\bar{\delta}_k(K) = \lim_{n_1,n_2,...,n_k} \sup \frac{1}{n_1n_2...n_k} \sum_{i_k=1}^{n_k} ... \sum_{i_2=1}^{n_2}\sum_{i_1=1}^{n_k} \chi_K(i_1,i_2,...,i_k) \text{ exists,}$$

where χ_K is the characteristic function of K.

Definition 8. A multiple sequence $x = (x_{n_1n_2....n_k})$ is said to be statistically convergent to L if for a given $\varepsilon > 0$,

$$\delta_k\left(\left\{\left(n_1,n_2,...,n_k\right) \in \mathbb{N}^k : |x_{n_1n_2....n_k} - L| \geq \varepsilon\right\}\right) = 0$$

and we write $st - \lim x_{n_1n_2....n_k} = L$.

Definition 9. A multiple sequence $x = (x_{n_1n_2....n_k})$ is said to be statistically null if for a given $\varepsilon > 0$,

$$\delta_k\left(\left\{\left(n_1,n_2,...,n_k\right) \in \mathbb{N}^k : |x_{n_1n_2....n_k}| \geq \varepsilon\right\}\right) = 0.$$

Definition 10. A multiple sequence $x = (x_{n_1n_2....n_k})$ is said to be statistically bounded if there exists a positive integer M such that,

$$\delta_k\left(\left\{\left(n_1,n_2,...,n_k\right) \in \mathbb{N}^k : |x_{n_1n_2....n_k}| > M\right\}\right) = 0.$$

Definition 11. A multiple sequence $x = (x_{n_1n_2....n_k})$ is said to be statistically convergent to $L \in X$ with respect to N if for every $\varepsilon > 0$ and $\beta \in (0,1)$,

$$\delta_k\left(\left\{\left(n_1,n_2,...,n_k\right) \in \mathbb{N}^k : N_{x_{n_1n_2....n_k}-L}(\varepsilon) \leq 1-\beta\right\}\right) = 0.$$

We write it as $st_N - \lim x_{n_1n_2....n_k} = L$.

Definition 12. A multiple sequence $x = (x_{n_1n_2....n_k})$ is statistically Cauchy with respect to N if for every $\varepsilon > 0$ and $\beta \in (0,1)$ there is a positive integer m_0 such that

$$\delta_k\left(\left\{\left(n_1,n_2,...,n_k\right) \in \mathbb{N}^k : N_{x_{n_1n_2....n_k}-x_{m_1m_2....m_k}}(\varepsilon) \leq 1-\beta\right\}\right) = 0.$$

Definition 13. Let $(X,N,*)$ be a probabilistic normed space. For $x \in X$, $t > 0$ and $0 < r < 1$, the ball centred at x with radius r is defined by

$$B(x,r,t) = \left\{y \in X : N_{x-y}(t) > 1-r\right\}.$$

Definition 14. A subset Y of $(X, N, *)$ is said to be bounded if for every $r \in (0,1)$ there exists $t_0 > 0$ such that

$$N_x(t_0) > 1 - r \text{ for all } x \in Y.$$

Definition 15. In a PN-space $(X, N, *)$, $L \in X$ is called a limit point of the multiple sequence $x = (x_{n_1 n_2 .... n_k})$ with respect to N if there is a subsequence of x that converges to L with respect to N. Let us denote the set of all limit points of the sequence x by $\Omega_N(x)$. If $(x_{n_1(j_1), n_2(j_2), ..., n_k(j_k)})$ is a subsequence of $x = (x_{n_1 n_2 .... n_k})$ and
$K = \Big\{ \Big(n_1(j_1), n_2(j_2), ..., n_k(j_k)\Big) \in \mathbb{N}^k : j_1, j_2, ..., j_k \in \mathbb{N} \Big\}$, then
$\Big\{ x_{n_1(j_1), n_2(j_2), ..., n_k(j_k)} \Big\}$ is abbreviated by $\Big\{ x \Big\}_K$. If $\delta_k(K) = 0$ then $\Big\{ x \Big\}_K$ is called a sub sequence of density zero or thin sub sequence. Also if $\delta_k(K) \neq 0$ then $\Big\{ x \Big\}_K$ is called a non-thin subsequence of x.

Definition 16. In a PN-space $(X, N, *)$, $\xi \in X$ is called a statistical limit point of the multiple sequence $x = (x_{n_1 n_2 .... n_k})$ with respect to N if there is a non-thin subsequence of x that converges to $\xi \in X$ with respect to N. ξ is called an $st_N - limit$ point of sequence $x = (x_{n_1 n_2 .... n_k})$. Let the set of all $st_N - limit$ points of the sequence x be denoted by $\Lambda_N(x)$.

Definition 17. In a PN-space $(X, N, *)$, $\gamma \in X$ is called a statistical cluster point of the sequence $x = (x_{n_1 n_2 .... n_k})$ with respect to N if for $\varepsilon > 0$ and $\beta \in (0,1)$,

$$\bar{\delta}_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - \gamma}(\varepsilon) > 1 - \beta \Big\} \Big) > 0.$$

γ is called an $st_N - cluster$ point of the sequence $x = (x_{n_1 n_2 .... n_k})$. Let the set of all $st_N - cluster$ points of the sequence x be denoted by $\Gamma_N(x)$.

Definition 18. A probabilistic normed space $(X, N, *)$ is said to be complete if every Cauchy sequence is convergent in X with respect to the probabilistic norm N.

Theorem 1. In a PN-space $(X, N, *)$, for every $\varepsilon > 0$ and $\beta \in (0,1)$, the following statements are equivalent.
(i) $st_N - \lim x_{n_1 n_2 .... n_k} = L$.
(ii) $\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big) = 0$.
(iii) $\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in N^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) > 1 - \beta \Big\} \Big) = 1$.
(iv) $st - \lim N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) = 1$.

Proof. $(i) \Rightarrow (ii)$
Suppose $st_N - \lim x_{n_1 n_2 .... n_k} = L$. Then by definition, for every $\varepsilon > 0$ and $\beta \in (0,1)$, we have
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big) = 0$.

$(ii) \Rightarrow (iii)$
Let $\varepsilon > 0$ and $\beta \in (0,1)$, then we have
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) > 1 - \beta \Big\} \Big)$.
$= 1 - \delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big)$.
$= 1$ by (ii).

$(iii) \Rightarrow (iv)$
Let $\varepsilon > 0$ and $\beta \in (0,1)$, then

$$\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : |N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) - 1| \geq \beta \Big\}$$

$$= \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\}$$
$$\cup \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \geq 1 + \beta \Big\}.$$

Therefore we have from the finite additivity property of density,
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : |N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) - 1| \geq \beta \Big\} \Big)$
$= \delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big)$
$+ \delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \geq 1 + \beta \Big\} \Big).$

Since, $\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big) = 0$
and
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \geq 1 + \beta \Big\} \Big) = 0.$
Hence
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : |N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) - 1| \geq \beta \Big\} \Big) = 0.$
Hence $st - \lim N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) = 1.$

$(iv) \Rightarrow (i)$
By hypothesis for a given $\varepsilon > 0$ and $\beta \in (0,1)$, we have
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : |N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) - 1| \geq \beta \Big\} \Big) = 0.$

i.e., $\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big)$
$+ \delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \geq 1 + \beta \Big\} \Big) = 0.$
$\Rightarrow \delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big) = 0,$
as
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \geq 1 + \beta \Big\} \Big) = 0.$

□

Theorem 2. In a PN-space $(X, N, *)$, if a sequence $x = (x_{n_1 n_2 .... n_k})$ is statistically convergent with respect to the probabilistic norm N, then $st_N - limit$ is unique.

Proof. We assume that $st_N - \lim x_{n_1 n_2 .... n_k} = M_1$ and $st_N - \lim x_{n_1 n_2 .... n_k} = M_2$ where $x = (x_{n_1 n_2 .... n_k})$ is a multiple sequence.

For a given $\lambda > 0$ we take $\beta \in (0,1)$ such that $(1 - \beta) * (1 - \beta) > 1 - \lambda$.
Then for given $\varepsilon > 0$, we define the following sets:
$K_{N,1}(\beta, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - M_1}(\varepsilon) \leq 1 - \beta \Big\},$

$K_{N,2}(\beta, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - M_2}(\varepsilon) \leq 1 - \beta \Big\}.$

Since $st_N - \lim x_{n_1 n_2 .... n_k} = M_1$, $\delta_k (\{K_{N,1}(\beta, \varepsilon)\}) = 0$, for all $\varepsilon > 0$.
Also, as $st_N - \lim x_{n_1 n_2 .... n_k} = M_2$, we get $\delta_k (\{K_{N,2}(\beta, \varepsilon)\}) = 0$, for all $\varepsilon > 0$.

Let $K_N(\beta,\varepsilon) = K_{N,1}(\beta,\varepsilon) \cap K_{N,2}(\beta,\varepsilon)$.
Then $\delta_k\left(\{K_N(\beta,\varepsilon)\}\right) = 0$ which implies that $\delta_k\left(\left\{\mathbb{N}^k / K_N(\beta,\varepsilon)\right\}\right) = 1$.

If $(n_1, n_2, ..., n_k) \in \left\{\mathbb{N}^k / K_N(\beta,\varepsilon)\right\}$, then

$$N_{M_1 - M_2}(\varepsilon) \geq N_{x_{n_1 n_2 ....n_k} - M_1}(\frac{\varepsilon}{2}) * N_{x_{n_1, n_2, ..., n_k} - M_2}(\frac{\varepsilon}{2}) > (1-\beta) * (1-\beta)$$
$$> 1 - \lambda.$$

Since $\lambda > 0$ is arbitrary, $N_{M_1 - M_2}(\varepsilon) = 1$ for all $\varepsilon > 0$. Thus $M_1 = M_2$. Therefore $st_N - limit$ of multiple sequence is unique.

□

Theorem 3. In a PN-space $(X, N, *)$, if $N - \lim x_{n_1 n_2 ....n_k} = M$, then $st - \lim x_{n_1 n_2 ....n_k} = M$, but the converse is not true.

Proof. By hypothesis $x = (x_{n_1 n_2 ....n_k})$, converges to M with respect to N. Therefore for every $\beta \in (0,1)$ and $\varepsilon > 0$ there is a positive integer m_0 such that
$N_{x_{n_1 n_2 \cdots n_k} - M}(\varepsilon) > 1 - \beta$ for all $n_i \geq m_0$, $i = 1, 2, 3, ..., k$.
Thus the set $\left\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \cdots n_k} - M}(\varepsilon) \leq 1 - \beta\right\}$ has finitely many terms.
Since every finite subset of $\mathbb{N}^k$ has density zero, we observe that
$\delta_k\left(\left\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \cdots n_k} - M}(\varepsilon) \leq 1 - \beta\right\}\right) = 0$.

□

The following example shows that the converse of Theorem 3 is not true.

Example 3. Let $(\mathbb{R}, |.|)$ denote the space of real numbers with usual norm. Let $a * b = ab$ and $N_x(t) = \frac{t}{t+|x|}$ where $x \in X$ and $t \geq 0$. Then $(\mathbb{R}, N, *)$ is a PN-space. We define a sequence $x = (x_{n_1 n_2 \ldots n_k})$ whose terms are given by

$$x_{n_1 n_2 \ldots n_k} = \begin{cases} 1, \ if\ n_1,\ n_2,\ ...,\ n_k \text{ are squares;} \\ 0, \ \text{otherwise.} \end{cases} \tag{2.1}$$

Then for every $\beta \in (0,1)$ and for any $\varepsilon > 0$,
let $K_N(\beta,\varepsilon) = \{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : \ N_{x_{n_1 n_2 \cdots n_k}}(\varepsilon) \leq 1 - \beta\}$.
Since

$$\begin{aligned} K_N(\beta,\varepsilon) &= \{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : \tfrac{t}{t+|x_{n_1 n_2 \cdots n_k}|} \leq 1 - \beta\} \\ &= \{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : |x_{n_1 n_2 \ldots n_k}| \geq \tfrac{\beta t}{1-\beta} > 0\} \\ &= \{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : x_{n_1 n_2 \ldots n_k} = 1\} \\ &= \{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : n_1,\ n_2, ...,\ n_k \ are\ squares\}, \end{aligned}$$

we get

$$\frac{1}{n_1 n_2 \ldots n_k} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \chi_K\left(i_1, i_2, ..., i_k\right) \leq \frac{\sqrt{n_1}\sqrt{n_2}...\sqrt{n_k}}{n_1 n_2 \ldots n_k}$$

which implies that
$\lim\limits_{n_1, n_2, ..., n_k} \frac{1}{n_1 n_2 ....n_k} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \chi_K\left(i_1, i_2, ..., i_k\right) = 0$. But the sequence $x = (x_{n_1 n_2 \ldots n_k})$ is not convergent in $(\mathbb{R}, |.|)$ with respect to the probabilistic norm N.

Following the technique applied by Salat [15] for establishing the decomposition result for statistical convergence for single sequences, we formulate the following result.

Theorem 4. In a PN-space $(X, N, *)$ and for a multiple sequence $x = (x_{n_1 n_2 \dots n_k})$, $st_N - \lim x_{n_1 n_2 \dots n_k} = L$ if and only if there exists an index subset $K = \Big\{ \Big(m_{n_1}, m_{n_2}, ..., m_{n_k} \Big) : m_{n_k} \in \mathbb{N} \Big\}$ of $\mathbb{N}^k$ such that $\delta_k(K) = 1$ and $N - \lim\limits_{(n_1, n_2, ..., n_k) \in K} x_{n_1 n_2 \dots n_k} = L$.

Proof. Suppose that $st_N - \lim x_{n_1 n_2 \dots n_k} = L$.
Now for every $\varepsilon > 0$ and $r \in \mathbb{N}$, let

$$K(r, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) \leq 1 - \frac{1}{r} \Big\} \tag{2.2}$$

$$M(r, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) > 1 - \tfrac{1}{r} \Big\}$$

Then $\delta_k \Big(\Big\{ K(r, \varepsilon) \Big\} \Big) = 0$ and

$$M(1, \varepsilon) \supset M(2, \varepsilon) \supset M(3, \varepsilon) \supset .... \supset M(i, \varepsilon) \supset M(i+1, \varepsilon) \supset .... \tag{2.3}$$

$$\delta_k \Big(\Big\{ M(r, \varepsilon) \Big\} \Big) = 1 \quad for \quad r = 1, 2, 3, ... \tag{2.4}$$

Now we have to show that for $(n_1, n_2, ..., n_k) \in M(r, \varepsilon)$, the sequence $x = (x_{n_1 n_2 \dots n_k})$ is $N-$ convergent to L.

Suppose that $x = (x_{n_1 n_2 \dots n_k})$ is not N-convergent to L. Therefore there exists $\beta > 0$ such that the set
$\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) \leq 1 - \beta \Big\}$ has infinitely many terms.
Let $M(\beta, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) > 1 - \beta \Big\}$, $\beta > \frac{1}{r}$, $(r = 1, 2, 3, ...)$.
Then $\delta_k \Big(\Big\{ M(\beta, \varepsilon) \Big\} \Big) = 0$ and by (2.3) we have $M(r, \varepsilon) \subset M(\beta, \varepsilon)$.
Hence $\delta_k \big(\{ M(r, \varepsilon) \} \big) = 0$ which contradicts (2.4).
Therefore $x = (x_{n_1 n_2 \dots n_k})$ is N-convergent to L.

Conversely let us suppose that there is a subset
$K = \Big\{ (n_1, n_2, ..., n_k) : n_i = 1, 2, 3, 4, ..., i \in \mathbb{N} \Big\} \subset \mathbb{N}^k$ such that $\delta_k(K) = 1$ and
$N - \lim\limits_{\substack{(n_1, n_2, ..., n_k) \in K \\ n_1, n_2, ..., n_k \to \infty}} x_{n_1 n_2 \dots n_k} = L$. Then there exists $k_0 \in \mathbb{N}$, such that for every $\beta \in (0, 1)$ and $\varepsilon > 0$,
$N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) > 1 - \beta$ for $n_i \geq k_0$, $i = 1, 2, 3, ...k$.

Now,
$\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) \leq 1 - \beta \Big\}$

$$\subset \mathbb{N}^k - \Big\{ \big(n_{1(k_0+1)}, n_{2(k_0+1)} ..., n_{k(k_0+1)} \big), \big(n_{1(k_0+2)}, n_{2(k_0+2)}, ..., n_{k(k_0+2)} \big), \big(n_{1(k_0+3)}, n_{2(k_0+3)}, ..., n_{k(k_0+3)} \big), ... \Big\}.$$

Therefore
$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \dots n_k} - L}(\varepsilon) \leq 1 - \beta \Big\} \Big) \leq 1 - 1 = 0.$
Hence $st_N - \lim x_{n_1 n_2 \dots n_k} = L$.

□

Theorem 5. In a PN-space $(X, N, *)$ and for a multiple sequence $x = (x_{n_1 n_2 \ldots. n_k})$ whose terms are in the vector space X, the following conditions are equivalent.
(a) X is statistically Cauchy sequence with respect to the probabilistic norm N.
(b) There exists an index subset $K = \Big\{ (m_{n_1}, m_{n_2}, ..., m_{n_k}) \Big\}$ of $\mathbb{N}^k$ such that $\delta_k(K) = 1$ and the subsequence $\Big\{ \Big(x_{m_{n_1} m_{n_2} \ldots m_{n_k}} \Big) \Big\}_{(m_{n_1}, m_{n_2}, \ldots, m_{n_k}) \in K}$ is a Cauchy sequence with respect to the probabilistic norm N.

Proof. The proof is similar to proof of Theorem 4 and thus omitted. □

Theorem 6. Let $(X, N, *)$ be a PN-space.Then
(i) If $st_N - \lim x_{n_1 n_2 \ldots. n_k} = \xi$ and $st_N - \lim y_{n_1 n_2 \ldots. n_k} = \eta$, then
$st_N - \lim (x_{n_1 n_2 \ldots. n_k} + y_{n_1 n_2 \ldots. n_k}) = \xi + \eta$.
(ii) If $st_N - \lim x_{n_1 n_2 \ldots. n_k} = \xi$ and $\alpha \in R$, then
$st_N - \lim \alpha x_{n_1 n_2 \ldots. n_k} = \alpha \xi$.
(iii) If $st_N - \lim x_{n_1 n_2 \ldots. n_k} = \xi$ and $st_N - \lim y_{n_1 n_2 \ldots. n_k} = \eta$, then
$st_N - \lim (x_{n_1 n_2 \ldots. n_k} - y_{n_1 n_2 \ldots. n_k}) = \xi - \eta$.

Proof. **(i)** Let $st_N - \lim x_{n_1 n_2 \ldots n_k} = \xi$ and $st_N - \lim y_{n_1 n_2 \ldots n_k} = \eta$.
For a given $\varepsilon > 0$ and $\lambda \in (0,1)$ we take $\beta \in (0,1)$ such that $(1-\beta) * (1-\beta) > 1 - \lambda$. We define the following sets.

$$K_{N,1}(\beta, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots n_k} - \xi}(\varepsilon) \leq 1 - \beta \Big\},$$
$$K_{N,2}(\beta, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 \ldots n_k} - \eta}(\varepsilon) \leq 1 - \beta \Big\}.$$

Since $st_N - \lim x_{n_1 n_2 \ldots n_k} = \xi$, $\delta_k \big(\{ K_{N,1}(\beta, \varepsilon) \} \big) = 0$, for all $\varepsilon > 0$.
Also as $st_N - \lim x_{n_1 n_2 \ldots n_k} = \eta$ we get $\delta_k \big(\{ K_{N,2}(\beta, \varepsilon) \} \big) = 0$, for all $\varepsilon > 0$.

Now let $K_N(\beta, \varepsilon) = K_{N,1}(\beta, \varepsilon) \cap K_{N,2}(\beta, \varepsilon)$.
Then $\delta_k \left(\{ K_N(\beta, \varepsilon) \} \right) = 0$, which gives $\delta_k \big(\{ \mathbb{N}^k / K_N(\beta, \varepsilon) \} \big) = 1$.

If $(n_1, n_2, ..., n_k) \in \Big\{ \mathbb{N}^k / K_N(\beta, \varepsilon) \Big\}$, then

$$N_{\left(x_{n_1 n_2 \ldots n_k} - \xi\right) + \left(y_{n_1 n_2 \ldots n_k} - \eta\right)}(\varepsilon) \geq N_{x_{n_1 n_2 \ldots n_k} - \xi}\Big(\frac{\varepsilon}{2}\Big) * N_{x_{n_1 n_2 \ldots n_k} - \eta}\Big(\frac{\varepsilon}{2}\Big)$$
$$> (1-\beta) * (1-\beta) > 1 - \lambda.$$

Thus,
$$\delta_k \Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{\left(x_{n_1 n_2 \ldots n_k} - \xi\right) + \left(y_{n_1 n_2 \ldots n_k} - \eta\right)}(\varepsilon) \leq 1 - \lambda \Big\} \Big) = 0.$$
So, $st_N - \lim \Big(x_{n_1 n_2 \ldots n_k} + y_{n_1 n_2 \ldots n_k} \Big) = \xi + \eta$.

(ii) Let $st_N - \lim x_{n_1 n_2 \ldots n_k} = \eta$, $\beta \in (0,1)$ and $\varepsilon > 0$. Let us take $\alpha = 0$.
Then,
$N_{0 x_{n_1 n_2 \ldots n_k} - 0\xi}(\varepsilon) = N_0(\varepsilon) = 1 > 1 - \beta$.
So, $N - \lim 0 x_{n_1 n_2 \ldots n_k} = 0$.

Then from Theorem 3 we have, $st_N - \lim 0x_{n_1n_2...n_k} = 0$.

Now let $\alpha \in \mathbb{R}(\alpha \neq 0)$. As $st_N - \lim x_{n_1n_2...n_k} = \eta$, we define the following set

$K_N(\beta, \varepsilon) = \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1n_2...n_k} - \xi}(\varepsilon) \leq 1 - \beta \Big\}$ then
$\delta_k\left(\{K_N(\beta, \varepsilon)\}\right) = 0$ for all $\varepsilon > 0$.
We have, $\delta_k\Big(\Big\{\mathbb{N}^k / K_N(\beta, \varepsilon)\Big\}\Big) = 1$.

If $(n_1, n_2, ..., n_k) \in \Big(\Big\{\mathbb{N}^k / K_N(\beta, \varepsilon)\Big\}\Big)$, then

$$\begin{aligned}
N_{\alpha x_{n_1n_2...n_k} - \alpha\xi}(\varepsilon) &= N_{x_{n_1n_2...n_k} - \xi}\left(\tfrac{\varepsilon}{|\alpha|}\right) \\
&\geq N_{x_{n_1n_2....n_k} - \xi}(\varepsilon) * N_0\left(\tfrac{\varepsilon}{|\alpha|} - \varepsilon\right) \\
&= N_{x_{n_1n_2....n_k} - \xi}(\varepsilon) * 1 \\
&= N_{x_{n_1n_2...n_k} - \xi}(\varepsilon) > 1 - \beta, \text{ for } \alpha \in \mathbb{R}(\alpha \neq 0).
\end{aligned}$$

Then, $\delta_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{\alpha x_{n_1n_2...n_k} - \alpha\xi}(\varepsilon) \leq 1 - \beta \Big\}\Big) = 0$.
Thus $st_N - \lim \alpha x_{n_1n_2...n_k} = \alpha\xi$.

(iii) Follows from (i) and (ii) by putting $\alpha = -1$.

□

Theorem 7. In a PN-space $(X, N, *)$, for any multiple sequence
$x = (x_{n_1n_2....n_k}) \in X$, $\Lambda_N(x) \subset \Gamma_N(x)$.

Proof. Let $\xi \in \Lambda_N(x)$, then there is a non-thin subsequence $\left(x_{n_1(j_1)n_2(j_2),...,n_k(j_k)}\right)$ of $x = (x_{n_1n_2....n_k})$ that converges to ξ with respect to N, i.e.,
$\delta_k\Big(\Big\{\big(n_1(j_1), n_2(j_2), ..., n_k(j_k)\big) \in \mathbb{N}^k : N_{x_{n_1(j_1)n_2(j_2)....n_k(j_k)} - \xi}(\varepsilon) > 1 - \beta\Big\}\Big) = d > 0$.

Since
$\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1n_2...n_k} - \xi}(\varepsilon) > 1 - \beta\Big\}$
$\supset \Big\{ (n_1(j_1), n_2(j_2), ..., n_k(j_k)) \in \mathbb{N}^k : N_{x_{n_1(j_1)n_2(j_2)....n_k(j_k)} - \xi}(\varepsilon) > 1 - \beta\Big\}$
for every $\varepsilon > 0$, we get
$\Big\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1n_2....n_k} - \xi}(\varepsilon) > 1 - \beta\Big\}$
$\supseteq \Big\{\big(n_1(j_1), n_2(j_2), ..., n_k(j_k)\big) \in \mathbb{N}^k : j_1, j_2, ..., j_k \in \mathbb{N}\Big\}$
$-\Big\{\big(n_1(j_1), n_2(j_2), ...n_k(j_k)\big) \in \mathbb{N}^k : N_{x_{n_1(j_1)n_2(j_2)...n_k(j_k)} - \xi}(\varepsilon) \leq 1 - \beta\Big\}$.
As $\Big(x_{n_1(j_1)n_2(j_2)...n_k(j_k)}\Big)$ converges to ξ with respect to N, the set
$\Big\{(n_1(j_1), n_2(j_2), ...n_k(j_k)) \in \mathbb{N}^k : N_{x_{n_1(j_1)n_2(j_2)...n_k(j_k)} - \xi}(\varepsilon) \leq 1 - \beta\Big\}$ is finite, for any $\varepsilon > 0$, therefore
$\bar{\delta}_k\Big(\Big\{(n_1, n_2, ...n_k) \in \mathbb{N}^k : N_{x_{n_1n_2...n_k} - \xi}(\varepsilon) > 1 - \beta\Big\}\Big)$
$\geq \bar{\delta}_k\Big(\Big\{(n_1(j_1), n_2(j_2), ..., n_k(j_k)) \in \mathbb{N}^k : j_1, j_2, ..., j_k \in \mathbb{N}\Big\}\Big)$
$-\bar{\delta}_k\Big(\Big\{(n_1(j_1), n_2(j_2), ..., n_k(j_k)) \in \mathbb{N}^k : N_{x_{n_1(j_1)n_2(j_2)...n_k(j_k)} - \xi}(\varepsilon) \leq 1 - \beta\Big\}\Big)$.

Hence $\bar{\delta}_k\Big(\Big\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - \xi}(\varepsilon) > 1-\beta\Big\}\Big) > 0$. This proves that $\xi \in \Gamma_N(x)$.
Thus $\Lambda_N(x) \subset \Gamma_N(x)$. □

Theorem 8. In a PN-space $(X, N, *)$, for any multiple sequence $x = (x_{n_1 n_2 .... n_k}) \in X$, $\Gamma_N(x) \subset \Omega_N(x)$.

Proof. Let $\gamma \in \Gamma_N(x)$, then for every $\varepsilon > 0$ and $\beta \in (0, 1)$,
$\delta_k\Big(\Big\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 ... n_k} - \gamma}(\varepsilon) > 1-\beta\Big\}\Big) > 0$.
We set $\Big\{x\Big\}_K$ be a non-thin subsequence of x such that
$K = \Big\{\big(n_1(j_1), n_2(j_2), ..., n_k(j_k)\big) \in \mathbb{N}^k : N_{x_{n_1(j_1) n_2(j_2) ... n_k(j_k)} - \gamma}(\varepsilon) > 1-\beta\Big\}$ for every $\varepsilon > 0$ and $\delta_k(K) \neq 0$.
Since K has infinitely many elements, so $\gamma \in \Omega_N(x)$.
Thus $\Gamma_{N\Delta}(x) \subset \Omega_{N\Delta}(x)$.

□

Theorem 9. In a PN-space $(X, N, *)$, for any multiple sequence $x = (x_{n_1 n_2 .... n_k}) \in X$, $st_N - \lim x = L$, implies $\Lambda_N(x) = \Gamma_N(x) = \{L\}$.

Proof. First we prove that $\Lambda_N(x) = \{L\}$.
Let $\Lambda_N(x) = \{L, M\}$ such that $L \neq M$. Then there are non-thin sub sequences $\big(x_{n_1(j_1) n_2(j_2) ... n_k(j_k)}\big)$ and $\big(x_{m_1(j_1) m_2(j_2) ... m_k(j_k)}\big)$ of $x = (x_{n_1 n_2 ... n_k})$ that converges to L and M respectively with respect to N.

As $\Big(x_{m_1(j_1) m_2(j_2) ... m_k(j_k)}\Big)$ converges to M with respect to N, so for every $\varepsilon > 0$ and $\beta \in (0, 1)$, the set
$K = \Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1) m_2(j_2) .... m_k(j_k)} - M}(\varepsilon) \leq 1-\beta\Big\}$ is a finite set. Thus $\delta_k(K) = 0$.
Then
$\Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : j_1, j_2, ..., j_k \in \mathbb{N}\Big\}$
$= \Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1) m_2(j_2) .... m_k(j_k)} - M}(\varepsilon) > 1-\beta\Big\}$
$\cup \Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1) m_2(j_2) .... m_k(j_k)} - M}(\varepsilon) \leq 1-\beta\Big\}$
which shows that

$$\delta_k\Big(\Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1) m_2(j_2) .... m_k(j_k)} - M}(\varepsilon) > 1-\beta\Big\}\Big) \neq 0. \quad (2.5)$$

Since $st_N - \lim x = L$,

$$\delta_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) \leq 1-\beta\Big\}\Big) = 0 \; for \; every \; \varepsilon > 0. \quad (2.6)$$

Therefore $\delta_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) > 1-\beta\Big\}\Big) \neq 0$.
For every $L \neq M$,
$\Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1) m_2(j_2) .... m_k(j_k)} - M}(\varepsilon) > 1-\beta\Big\}$
$\cap \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k} - L}(\varepsilon) > 1-\beta\Big\} = \emptyset$.

Hence
$\Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1)m_2(j_2)....m_k(j_k)}-M}(\varepsilon) > 1-\beta\Big\}$
$\subseteq \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-L}(\varepsilon) \leq 1-\beta\Big\}.$

Therefore
$\bar{\delta}_k\Big(\Big\{\big(m_1(j_1), m_2(j_2), ..., m_k(j_k)\big) \in \mathbb{N}^k : N_{x_{m_1(j_1)m_2(j_2)....m_k(j_k)}-M}(\varepsilon) > 1-\beta\Big\}\Big)$
$\leq \bar{\delta}_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-L}(\varepsilon) \leq 1-\beta\Big\}\Big) = 0.$
This contradicts (2.5) Hence $\Lambda_N(x) = \{L\}$.

Next we show that $\Gamma_N(x) = \{L\}$.
If possible let $\Gamma_N(x) = \{L, Q\}$ such that $L \neq Q$. Then

$$\bar{\delta}_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-Q}(\varepsilon) > 1-\beta\Big\}\Big) \neq 0. \tag{2.7}$$

Since $\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-L}(\varepsilon) > 1-\beta\Big\}$
$\cap\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-Q}(\varepsilon) > 1-\beta\Big\} = \emptyset$, for every $L \neq Q$, so
$\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-L}(\varepsilon) \leq 1-\beta\Big\}$
$\supseteq \Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-Q}(\varepsilon) > 1-\beta\Big\}.$

Therefore

$$\begin{aligned}&\bar{\delta}_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-L}(\varepsilon) \leq 1-\beta\Big\}\Big)\\ &\geq \bar{\delta}_k\Big(\Big\{ (n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{x_{n_1 n_2 .... n_k}-Q}(\varepsilon) > 1-\beta\Big\}\Big).\end{aligned} \tag{2.8}$$

From (2.7), the right hand side of (2.8) is greater than zero. Also from (2.6) the left hand side of (2.8) equals zero which is a contradiction. Hence $\Gamma_N(x) = \{L\}$.

□

Theorem 10. In a PN-space $(X, N, *)$, the set Γ_N is closed in X for each multiple sequence $x = (x_{n_1 n_2 .... n_k})$ of elements of X.

Proof. Let $y \in \overline{\Gamma_N(x)}$. Let $0 < r < 1$ and $t > 0$. There exists
$\gamma \in \Gamma_N(x) \cap B(y, r, t)$ such that
$B(y, r, t) = \Big\{x \in X : N_{y-x}(t) > 1-r\Big\}.$

Choose $\eta > 0$ such that $B(\gamma, \eta, t) \subset B(y, r, t)$, then we have
$\Big\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{y-x_{n_1 n_2 ... n_k}}(t) > 1-r\Big\}$
$\supset \Big\{(n_1, n_2, .., n_k) \in \mathbb{N}^k : N_{\gamma-x_{n_1 n_2 .... n_k}}(t) > 1-\eta\Big\}.$

Since $\gamma \in \Gamma_N(x)$ so,
$\bar{\delta}_k\Big(\Big\{(n_1, n_2, .., n_k) \in \mathbb{N}^k : N_{\gamma-x_{n_1 n_2 .... n_k}}(t) > 1-\eta\Big\}\Big) > 0.$

Hence
$\bar{\delta}_k\Big(\Big\{(n_1, n_2, ..., n_k) \in \mathbb{N}^k : N_{y-x_{n_1 n_2 ... n_k}}(t) > 1-r\Big\}\Big) > 0.$

□

REFERENCES

[1] *M. Basarir and O. Solancan*,On some double sequence spaces, J. Indian Acad. Math., **21**(1999), No. 2, 193–200.

[2] *TJ. Bromwich*, An Introduction to the Theory of Infinite Series, McMillan, London., (1965).

[3] *G. Constantion and I. Istratescu*, Elements of probabilistic analysis, (1989).

[4] *H. Fast*, Sur la convergence statistique, Colloquium Mathematicae, **2**(1951), No. 3-4, 241–244.

[5] *M. Fréchet*, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo, **22**(1906), No. 1, 1–72.

[6] *G. H. Hardy*, On the convergence of certain multiple series, Proceedings of the cambridge Philosophical Society, **19**(1917), No. 3, 86–95.

[7] *S. Karakus*, Statistical convergence on probalistic normed spaces, Mathematical Communications, **12**(2007), No.1, 11–23.

[8] *S. Karakus and K. Demırcı*, Statistical convergence of double sequences on probabilistic normed spaces, International Journal of Mathematics and Mathematical Sciences, volume2007, year2007.

[9] *K. Menger*, Géométrie générale,1954.

[10] , *K. Menger*, Probabilistic geometry, Proceedings of the National Academy of Sciences of the United States of America,**37**(1951), No.4, pages226.

[11] *K. Menger*, Statistical metrics, Proceedings of the National Academy of Sciences of the United States of America, **28**(1942), No. 12.

[12] *F. Moricz*, Extensions of the spaces c and c_0 from single to double sequences, Acta Mathematica Hungarica, **57**(1991), No. 1, 129–136.

[13] *F. Moricz and BE. Rhoades*, Almost convergence of double sequences and strong regularity of summability matrices, Mathematical Proceedings of the Cambridge Philosophical Society, **104**(1988), No. 2, 283–294.

[14] *A. Pringsheim*, Zur theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen., **53**(1900), No. 3, 289-321.

[15] *T. Šalát*, On statistically convergent sequences of real numbers, Mathematica Slovaca, **30**(1980), No. 2, 139–150.

[16] *IJ Schoenberg*, The integrability of certain functions and related summability methods, American Mathematical Monthly, 361–375, 1959.

[17] *B. Schweizer, A. Sklar and others*, Statistical metric spaces, Pacific J. Math., **10**(1960), No. 1, 313-335.

[18] *B. Schweizer, A. Sklar*, Probabilistic Metric Spaces, North-Holland, New York, 1983.

[19] *B. C. Tripathy*, Statistically convergent double sequences, Tamkang Journal of Mathematics, **34**(2003), No. 3, 231–237.

[20] *B. C. Tripathy and A. Baruah*, New type of difference sequence spaces of fuzzy real numbers, Mathematical Modelling and Analysis, **14**(2009), No. 3, 391–397.

[21] *B. C. Tripathy and A. Baruah*, Nörlund and Riesz mean of sequences of fuzzy real numbers, Applied Mathematics Letters, **23**(2010), No. 5, 651–655.

[22] *B. C. Tripathy and S. Borgohain*, Difference sequence space $m(M, \Phi, \Delta_m^n, p)_F$ of fuzzy real numbers, Mathematical Modelling and Analysis, **13**(2008), No. 4, 577–586.

[23] *B. C. Tripathy and S. Borgohain*, Some classes of difference sequence spaces of fuzzy real numbers defined by Orlicz function, Advances in Fuzzy systems, year 2011.

[24] *B. C. Tripathy and A. J. Dutta*, On fuzzy real-valued double sequence space, Mathematical and Computer Modelling, **46**(2007), No. 9, 1294–1299.

[25] *B. C. Tripathy and B. Sarma*, Statistically convergent difference double sequence spaces, Acta Mathematica Sinica, English Series, **24**(2008), No. 5, 737–742.

[26] *B. C. Tripathy and B. Sarma*, Sequence spaces of fuzzy real numbers defined by Orlicz functions, Mathematica Slovaca, 5**8**(2008), No. 5, 621–628.

[27] *B. C. Tripathy and B. Sarma*, Vector valued double sequence spaces defined by Orlicz function, Mathematica Slovaca, **59**(2009), No. 6, 767–776.

[28] *B. C. Tripathy, M. Sen and S. Nath*, $\mathfrak{I}$-convergence in probabilistic n-normed space, Soft Computing, **16**(2012), No. 6, 1021–1027.

[29] *A. Wald*, On a statistical generalization of metric spaces, Proceedings of the National Academy of Sciences of the United States of America, **29**(1943), No. 6, pages196.