FIBONACCI BALANCING NUMBERS

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In memoriam Péter Kiss

ABSTRACT

A positive integer n is called a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some natural number r. We prove that there is no Fibonacci balancing number except 1.

1. INTRODUCTION

The sequence $\{R_n\}_{n=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence if the recurrence relation

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1)$$

holds for its terms, where A, $B \neq 0$, R_0 and R_1 are fixed rational integers and $|R_0| + |R_1| > 0$. The polynomial $x^2 - Ax - B$ is called the companion polynomial of the second order linear recurrence sequence $R = R(A, B, R_0, R_1)$. The zeros of the companion polynomial will be denoted by α and β. Using this notation, as it is well known, we get

$$R_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta}, \tag{1}$$

where $a = R_1 - R_0\beta$ and $b = R_1 - R_0\alpha$ (see [6]).

A positive integer n is called a *balancing number* [3] if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some $r \in \mathbf{Z}_+$. Here r is called the *balancer* corresponding to the balancing number n. For example 6 and 35 are balancing numbers with balancers 2 and 14. In a joint paper A. Behera and G. K. Panda [3] proved that the balancing numbers fulfil the following recurrence relation

$$B_{n+1} = 6B_n - B_{n-1} \quad (n > 1) \tag{2}$$

where $B_0 = 1$ and $B_1 = 6$.

We call a balancing number a *Fibonacci balancing number* if it is a Fibonacci number, too. In the next section we prove that there are no Fibonacci balancing numbers.

2. FIBONACCI BALANCING NUMBERS

The equation $x^2 - Dy^2 = N$ with given integers D and N and variables x and y, is called Pell's equation. First we prove that the balancing numbers are solutions of a Pell's equation.

Theorem 1: The terms of the second order linear recurrence $B(6,-1,1,6)$ are the solutions of the equation

$$z^2 - 8y^2 = 1 \tag{3}$$

for some integer z.

Proof: Let $B(6,-1,1,6)$ be a second order linear recurrence and denote by α and β the zeros of their companion polynomials and D the discriminant of the companion polynomial. Using (1) and the definition of α and β we get

$$B_n = \frac{(3+2\sqrt{2})\alpha^n - (3-2\sqrt{2})\beta^n}{4\sqrt{2}} \quad \text{and} \quad \alpha\beta = 1,$$

therefore, with $y = B_n$, we have

$$1 + 8y^2 = 1 + 8B_n^2 = 1 + \frac{8}{32}\Big((3+2\sqrt{2})^2\alpha^{2n}$$

$$-2(3+2\sqrt{2})(3-2\sqrt{2})\alpha^n\beta^n + (3-2\sqrt{2})^2\beta^{2n}\Big)$$

$$= 1 + \left(\frac{(3+2\sqrt{2})\alpha^n}{2}\right)^2 - \frac{1}{2} + \left(\frac{(3-2\sqrt{2})\beta^n}{2}\right)^2$$

$$= \left(\frac{(3+2\sqrt{2})\alpha^n}{2} + \frac{(3-2\sqrt{2})\beta^n}{2}\right)^2 = z^2.$$

Using that $\alpha = 3+2\sqrt{2}, \beta = 3-2\sqrt{2}$ and the binomial formula it can be proved that z is a rational integer.

To prove our main result we need the following theorem of P. E. Ferguson [4].

Theorem 2: The only solutions of the equation

$$x^2 - 5y^2 = \pm 4 \tag{4}$$

are $x = \pm L_n, y = \pm F_n$ $(n = 0, 1, 2, \dots)$, where L_n and F_n are the n^{th} terms of the Lucas and Fibonacci sequences, respectively.

Using the method of A. Baker and H. Davenport we prove that there are finitely many common solutions of the Pell's equations (3) and (4). We remark that this result follows from a theorem of P. Kiss [5], too. In the process we show that there are no Fibonacci balancing numbers. In the proof we use the following theorem of A. Baker and H. Wüstholz [2].

Theorem 3: Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers not 0 or 1, and let

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

where $b_1, \dots, b_n$ are rational integers not all zeros.

We suppose that $B = \max(|b_1|, \dots, |b_k|, e)$ and $A_i = \max\{(H(\alpha_i), e\} \quad (i = 1, 2, \dots n)$. Assume that the field K generated by $\alpha_1, \alpha_2, \dots, \alpha_n$ over the rationals has degree at most d. If $\Lambda \neq 0$ then

$$\log|\Lambda| > -(16nd)^{2(n+2)} \log A_1 \log A_2 \dots \log A_n \log B.$$

($H(\alpha)$ is equal to the maximum of absolute values of the coefficients of the minimal defining polynomial of α.)

The following theorem is the main result of this paper.

Theorem 4: There is no Fibonacci balancing number except 1.

Proof: First we show that there are finitely many common solutions of the equations (5), (6) and (5′), (6′)

$$5x^2 + 4 = y^2 \qquad (5) \qquad\qquad 5x^2 - 4 = y^2 \qquad (5')$$

$$8x^2 + 1 = z^2 \qquad (6) \qquad\qquad 8x^2 + 1 = z^2 \qquad (6')$$

The equations (5) and (5′) can be written as

$$(y + x\sqrt{5})(y - x\sqrt{5}) = 4 \tag{7}$$

and

$$(y + x\sqrt{5})(y - x\sqrt{5}) = -4. \tag{8}$$

If we put

$$y + x\sqrt{5} = (y_0 + x_0\sqrt{5})(9 + 4\sqrt{5})^m$$

where $m \geq 0$, it is easily verified (by combining this equation with its conjugate) that y_0 is always positive but x_0 is negative if m is large. Hence we can choose m so that $x_0 > 0$; but if x_1 is defined by

$$y_0 + x_0\sqrt{5} = (y_1 + x_1\sqrt{5})(9 + 4\sqrt{5})$$

then $x_1 \leq 0$. Since

$$y_0 + x_0\sqrt{5} = (9y_1 + 20x_1) + (9x_1 + 4y_1)\sqrt{5}$$

we have $y_0 = 9y_1 + 20x_1$ and $x_0 = 9x_1 + 4y_1$. From the previous equations we have $x_1 = 9x_0 - 4y_0$ and $x_0 \leq \frac{4y_0}{9}$. Using equation (5) we have

$$y_0^2 - 4 = 5x_0^2 \leq \frac{80}{81}y_0^2.$$

Hence $y_0 = 3, 7, 18$ and $x_0 = 1, 3, 8$, respectively. Thus the general solution of equation (5) is given by

$$y + x\sqrt{5} = (3 + \sqrt{5})(9 + 4\sqrt{5})^m \tag{9}$$

$$y + x\sqrt{5} = (7 + 3\sqrt{5})(9 + 4\sqrt{5})^m \tag{10}$$

$$y + x\sqrt{5} = (18 + 8\sqrt{5})(9 + 4\sqrt{5})^m \tag{11}$$

where $m = 0, 1, 2 \dots$

Using the same method as before with ($5'$), we find that $y_1 = 9y_0 - 20x_0 \le 0$ (in this case x_0 is always positive), whence $y_0^2 = 5x_0^2 - 4 \le \frac{400}{81}x_0^2$, so that $x_0 = 1, 2, 5$ and $y_0 = 1, 4, 11$, respectively. Thus the general solution of equation ($5'$) is given by

$$y + x\sqrt{5} = (1+\sqrt{5})(9+4\sqrt{5})^m \tag{12}$$

$$y + x\sqrt{5} = (4+2\sqrt{5})(9+4\sqrt{5})^m \tag{13}$$

$$y + x\sqrt{5} = (11+5\sqrt{5})(9+4\sqrt{5})^m \tag{14}$$

where $m = 0, 1, 2, \dots$

The general solution of equations (6) and ($6'$) is given by

$$z + \sqrt{8}x = (3+\sqrt{8})^n \tag{15}$$

where $n = 0, 1, 2, \dots$ We are looking for the common solutions of the equation (9), (10), (11), (12), (13), (14) with the equations (15). Using (9), (15) and their conjugates we have

$$2x = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3-\sqrt{8})^n}{\sqrt{8}} = \frac{3+\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{3-\sqrt{5}}{\sqrt{5}}(9-4\sqrt{5})^m$$

and so

$$\begin{gathered} \frac{1}{\sqrt{8}}(3+\sqrt{8})^n - \frac{(3+\sqrt{8})^{-n}}{\sqrt{8}} = \\ \frac{\sqrt{5}+3}{\sqrt{5}}(9+4\sqrt{5})^m + \frac{\sqrt{5}-3}{\sqrt{5}}(9+4\sqrt{5})^{-m}. \end{gathered} \tag{16}$$

Putting

$$Q = \frac{1}{\sqrt{8}}(3+\sqrt{8})^n, \quad P = \frac{\sqrt{5}+3}{\sqrt{5}}(9+4\sqrt{5})^m,$$

in equation (16) we obtain

$$Q - \frac{1}{8}Q^{-1} = P - \frac{4}{5}P^{-1}. \tag{17}$$

Since

$$Q - P = \frac{1}{8}Q^{-1} - \frac{4}{5}P^{-1} < \frac{4}{5}(Q^{-1} - P^{-1}) = \frac{4}{5}\frac{P-Q}{QP}$$

and plainly $P > 1$ and $Q > 1$, we have $Q < P$. Also $P - Q = \frac{4}{5}P^{-1} - \frac{1}{8}Q^{-1} < \frac{4}{5}P^{-1}$ and $P > 20$.

It follows that

$$0 < \log \frac{P}{Q} = -\log\left(1 - \frac{P-Q}{P}\right) < \frac{4}{5}P^{-2} + \left(\frac{4}{5}P^{-2}\right)^2 =$$

$$\frac{4}{5}P^{-2} + \frac{16}{25}P^{-4} < 0.81P^{-2} < \frac{0.15}{(9+4\sqrt{5})^{2m}}.$$

Using the previous inequality and the definitions of P and Q, we get

$$0 < m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log\frac{(3+\sqrt{5})\sqrt{8}}{\sqrt{5}} < \frac{0.15}{(9+4\sqrt{5})^{2m}}. \tag{18}$$

We apply Theorem 3 with $n = 3$ and

$$\alpha_1 = 9+4\sqrt{5} \qquad \alpha_2 = 3+\sqrt{8} \qquad \alpha_3 = \frac{(3+\sqrt{5})\sqrt{8}}{\sqrt{5}}.$$

We use that $0.15((9+4\sqrt{5})^2)^{-m} < e^{-5.77m}$. The equations satisfied by $\alpha_1, \alpha_2, \alpha_3$ are

$$\alpha_1^2 - 18\alpha_1 + 1 = 0 \qquad \alpha_2^2 - 6\alpha_2 + 1 = 0 \qquad 25\alpha_3^4 - 1120\alpha_3^2 + 1024 = 0.$$

Hence $A_1 = 18, A_2 = 6, A_3 = 1120$ and $d = 4$. Using Theorem 3 and the previous inequality we have

$$m < \frac{1}{5.77}(16\times 3\times 4)^{10}\log 18\log 6\log 1120\log m < 10^{24}\log m.$$

Thus we have

$$m < 10^{26}.$$

Using the same method we investigate the equations (10) and (15). We have

$$2x = \frac{3\sqrt{5}+7}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{7-3\sqrt{5}}{\sqrt{5}}(9-4\sqrt{5})^m = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3-\sqrt{8})^n}{\sqrt{8}}$$

that is

$$\frac{3\sqrt{5}+7}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{7-3\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^{-m} = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3+\sqrt{8})^{-n}}{\sqrt{8}}. \tag{19}$$

If we put

$$P_1 = \frac{7+3\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m, \qquad Q = \frac{1}{\sqrt{8}}(3+\sqrt{8})^n \tag{20}$$

using (19) and (20) we have

$$Q - \frac{1}{8}Q^{-1} = P_1 - \frac{4}{5}P_1^{-1}.$$

Using the previous method we have

$$0 < \log\frac{P_1}{Q} < 0.81P_1^{-2} < \frac{0.022}{(9+4\sqrt{5})^{2m}}.$$

Substituting from (20), we obtain

$$0 < m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log\frac{(7+3\sqrt{5})\sqrt{8}}{\sqrt{5}} < \frac{0.022}{(9+4\sqrt{5})^{2m}}. \tag{21}$$

We apply Theorem 3 with $n = 3$ and

$$\alpha_1 = 9+4\sqrt{5} \quad \alpha_2 = 3+\sqrt{8} \quad \alpha_3 = \frac{(7+3\sqrt{5})\sqrt{8}}{\sqrt{5}}.$$

The equation

$$25\alpha_3^4 - 7520\alpha_3^2 + 1024 = 0$$

is satisfied by α_3, that is $A_3 = 7520$. Using Theorem 3 as above we have

$$m < \frac{1}{5.77}(16\times 3\times 4)^{10}\log 18\log 6\log 7520\log m < 10^{24}\log m.$$

It follows that

$$m < 10^{26}.$$

From the equations (11) and (15) we have

$$2x = \frac{18+8\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{18-8\sqrt{5}}{\sqrt{5}}(9-4\sqrt{5})^m = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3-\sqrt{8})^n}{\sqrt{8}}$$

that is

$$\frac{18+8\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{18-8\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^{-m}$$

$$= \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3+\sqrt{8})^{-n}}{\sqrt{8}}. \tag{22}$$

If we put

$$P_2 = \frac{18+8\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m, \quad Q = \frac{(3+\sqrt{8})^n}{\sqrt{8}}. \tag{23}$$

Using (22) and (23) we have

$$Q - \frac{1}{8}Q^{-1} = P_2 - \frac{4}{5}P_2^{-1}.$$

As before we obtain

$$0 < \log\frac{P_2}{Q} = -\log\left(1 - \frac{P_2 - Q}{P_2}\right) < \frac{0.004}{(9+4\sqrt{5})^{2m}}.$$

Substituting from (23) we have

$$0 < m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log\frac{(18+8\sqrt{5})\sqrt{8}}{\sqrt{5}} < \frac{0.004}{(9+4\sqrt{5})^{2m}}. \tag{24}$$

We apply Theorem 3 as above with $n = 3$ and

$$\alpha_1 = 9+4\sqrt{5} \qquad \alpha_2 = 3+\sqrt{8} \qquad \alpha_3 = \frac{(18+8\sqrt{5})\sqrt{8}}{\sqrt{5}}.$$

The equation

$$25\alpha^4 - 51520\alpha^2 + 1024 = 0$$

is satisfied by α_3 and so $A_3 = 51520$.

$$m < \frac{1}{5.77}(16\times 3\times 4)^{10}\log 18\log 6\log 51520\log m < 10^{24}\log m.$$

It follows that

$$m < 10^{26}.$$

¿From the equations (12) and (15) we have

$$2x = \frac{1+\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{1-\sqrt{5}}{\sqrt{5}}(9-4\sqrt{5})^m = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3-\sqrt{8})^n}{\sqrt{8}}$$

that is

$$\frac{1+\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{1-\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^{-m} = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3+\sqrt{8})^{-n}}{\sqrt{8}}. \tag{25}$$

If we put

$$P_3 = \frac{1+\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m, \qquad Q = \frac{(3+\sqrt{8})^n}{\sqrt{8}}. \tag{26}$$

Using (25) and (26) we have

$$Q - \frac{1}{8}Q^{-1} = P_3 + \frac{4}{5}P_3^{-1}.$$

From the previous equation we have

$$Q - P_3 = \frac{4}{5}P_3^{-1} + \frac{1}{8}Q^{-1} > 0$$

that is $Q > P_3$ and

$$Q - P_3 < \frac{4}{5}P_3^{-1} + \frac{1}{8}P_3^{-1} = \frac{37}{40}P_3^{-1}.$$

It follows that

$$0 < \log \frac{Q}{P_3} = \log\left(1 + \frac{Q - P_3}{P_3}\right) < \frac{37}{40}P_3^{-2}$$

$$+ \left(\frac{37}{40}P_3^{-2}\right)^2 < 0.926P_3^{-2} < 0.443\frac{1}{(9+4\sqrt{5})^{2m}}$$

and so

$$0 < m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log\frac{\sqrt{5}}{(1+\sqrt{5})\sqrt{8}} < \frac{0.443}{(9+4\sqrt{5})^{2m}}. \tag{27}$$

Using the previous method and that the equation

$$1024\alpha^4 - 480\alpha^2 + 25 = 0$$

is satisfied by $\alpha_3 = \frac{\sqrt{5}}{(1+\sqrt{5})\sqrt{8}}$ and so $A_3 = 1024$. It follows that

$$m < \frac{1}{5.77}(16 \times 3 \times 4)^{10} \log 18 \log 6 \log 1024 \log m < 10^{24} \log m.$$

It follows that

$$m < 10^{26}.$$

¿From the equations (13) and (15) we have

$$\frac{4+2\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{4-2\sqrt{5}}{\sqrt{5}}(9-4\sqrt{5})^m = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3-\sqrt{8})^n}{\sqrt{8}}.$$

If we put

$$P_4 = \frac{4+2\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m, \qquad Q = \frac{(3+\sqrt{8})^n}{\sqrt{8}}$$

we get similar inequalities as before. We have

$$0 < \log\frac{Q}{P_4} < 0.926P_4^{-2} < 0.065\frac{1}{(9+4\sqrt{5})^{2m}}$$

and

$$0 < m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log\frac{\sqrt{5}}{(4+2\sqrt{5})\sqrt{8}} < 0.065\frac{1}{(9+4\sqrt{5})^{2m}}. \tag{28}$$

The equation

$$62464\alpha^4 - 2880\alpha^2 + 5 = 0$$

is satisfied by $\alpha_3 = \frac{\sqrt{5}}{(4+2\sqrt{5})\sqrt{8}}$ and so $A_3 = 62464$. We use the Theorem of A. Baker and G. Wüstholz again and we have

$$m < \frac{1}{5.77}(16\times 3\times 4)^{10}\log 18\log 6\log 62464\log m < 10^{24}\log m.$$

It follows that

$$m < 10^{26}.$$

Finally let's consider the equations (14) and (15). We have

$$\frac{11+5\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m - \frac{11-5\sqrt{5}}{\sqrt{5}}(9-4\sqrt{5})^m = \frac{(3+\sqrt{8})^n}{\sqrt{8}} - \frac{(3-\sqrt{8})^n}{\sqrt{8}}.$$

If we put

$$P_5 = \frac{11+5\sqrt{5}}{\sqrt{5}}(9+4\sqrt{5})^m, \quad Q = \frac{(3+\sqrt{8})^n}{\sqrt{8}}$$

and we use the previous steps we have

$$0 < \log\frac{Q}{P_5} < 0.926P_5^{-2} < 0.0095\frac{1}{(9+4\sqrt{5})^{2m}}$$

and

$$0 < m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log\frac{\sqrt{5}}{(11+5\sqrt{5})\sqrt{8}} < \frac{0.0095}{(9+4\sqrt{5})^{2m}}. \tag{29}$$

The equation

$$1024\alpha^4 - 19680\alpha^2 + 25 = 0$$

is satisfied by $\alpha_3 = \frac{\sqrt{5}}{(11+5\sqrt{5})\sqrt{8}}$ and so $A_3 = 19680$. We apply again Theorem 3 and we have

$$m < \frac{1}{5.77}(16 \times 3 \times 4)^{10} \log 18 \log 6 \log 19680 \log m < 10^{24} \log m.$$

It follows that

$$m < 10^{26}.$$

We get that there are finitely many common solutions of simultaneous equations, that is, there are finitely many Fibonacci balancing numbers. Since the bounds for m are too high ($m < 10^{26}$) we can't investigate all of them. In order to get a lower bound we use the following lemma of A. Baker and H. Davenport [1].

Lemma: Suppose that $K > 6$. For any positive integer M, let p and q be integers satisfying

$$1 \leq q \leq KM, \qquad |\theta q - p| < 2(KM)^{-1}.$$

Then, if $\|q\beta\| \geq 3K^{-1}$, there is no solution of the equation

$$|m\theta - n + \beta| < C^{-m}$$

in the range

$$\frac{\log K^2 M}{\log C} < m < M.$$

(It is supposed that θ, β are real numbers and $C > 1$. $\|z\|$ denotes the distance of a real number z from the nearest integer.)

We divide the inequality (18) by $\log(3+\sqrt{8})$ and we show the steps of reduction. In the other cases, (21), (24), (27), (28) and (29), the method is similar. Using the lemma we have

$$C = (9 + 4\sqrt{5})^2 = 321.997\ldots, \quad \theta = \frac{\log(9+4\sqrt{5})}{\log(3+\sqrt{8})}$$

and

$$\beta = \log \frac{(3+\sqrt{5})\sqrt{8}}{\sqrt{5}} (\log(3+\sqrt{8}))^{-1}.$$

In our case we take $M = 10^{26}, K = 100$. Let θ_0 be the value of θ correct to 56 decimal places, so that

$$|\theta - \theta_0| < 10^{-56}.$$

Let $\frac{p}{q}$ be the last convergent to the continued fraction for θ_0 which satisfies $q < 10^{28}$; then $|q\theta_0 - p| < 10^{-28}$. We therefore have

$$|q\theta - p| \leq q|\theta - \theta_0| + |q\theta_0 - p| < 2 \times 10^{-28}.$$

In this case the first and the second inequalities of the Lemma are satisfied. The values of θ, β and q computed by Maple are given in Appendix. We have that $\|q\beta\| = 0.4049\ldots$. It follows from the lemma, since $\|q\beta\| \geq 0.03$, there is no solution of (18) in the range

$$\frac{\log 10^{30}}{\log 321{,}997} < m < 10^{26}.$$

That is m less than 12, so we can calculate by hand that there is no Fibonacci balancing number in this case. We get the same result in the other cases, too.

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APPENDIX

$\theta = 1.637938209676347011669771024581365228556275262867141682518 88$

$q = 785070494894485057772397828 2$

$\beta = 0.68380257009531653018864575560326411558399742942127716547460 4$

$\|q\beta\| = 0.40491601596865450151807911061885$

$\beta_1 = 0.9279719820806909120343489025866123301668858551364546280670 58$

$\|q\beta_1\| = 0.37968838470510105521228687874339$

$\beta_2 = 1.17214139406606529388005204956996054474977428085163209065951$

$\|q\beta_2\| = 0.16429278537885661194265286810565$

$\beta_3 = 0.15497856294145147944094097030575025332816937135992515949828 3$

$\|q\beta_3\| = 0.33226779145594849106219897740840$

$\beta_4 = 0.05919654351601958516161144829363093544299849744818452885417 85$

$\|q\beta_4\| = 0.43056985568191587933854929140067 3$

$\beta_5 = 0.02261106760660727604389337457514255300082612098462842706425 26$

$\|q\beta_5\| = 0.37602264149830387092215314838958 5$

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