Orbiting Vertex: Follow That Triangle Center!

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1 Preliminaries

1.1 Introduction

The number of triangle centers is astounding. Upwards of eleven, hundred special triangle centers have been found. Almost all of these centers locations relate directly to the position of the vertices of the triangle that forms them. So moving the vertex must move the triangle centers in question. Last year, a student research group investigated what happened to some of these centers as one vertex of the triangle moved around a circle. In their paper *Tracing a Point, Spinning a Vertex: How Circles are Made*, [1] they found that the incenter, excenter, and orthocenter of a triangle trace a circle as one vertex of that triangle moves around a second circle while the other two vertices remain fixed on that second circle. This investigation aims to generalize their findings to ellipses as well as expand them to cover an additional triangle center, the centroid.

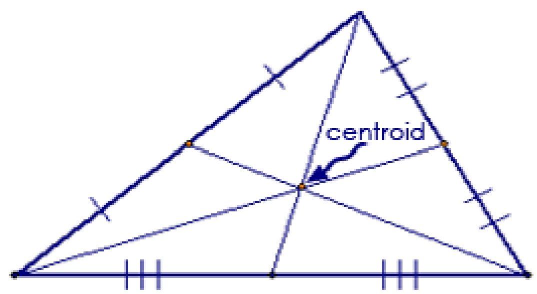
All of the results that follow begin with a similar set-up, and so it will be helpful from the outset to explain that set-up, rather than do so at the beginning of each theorem. All of the triangles described herein will have two points (defining the base) fixed at a specified position (not necessarily on an ellipse). The third point of the triangle will lie at an arbitrary point of an ellipse and so can be thought of as tracing the ellipse as different points are chosen. This ellipse, e_1, will be centered at the origin and have equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a is the length of the semi-major axis and b is the length of the semi-minor axis.

1.2 Constructions

The following is a list of terms with definitions that are used throughout the paper and pictures illustrating their constructions.

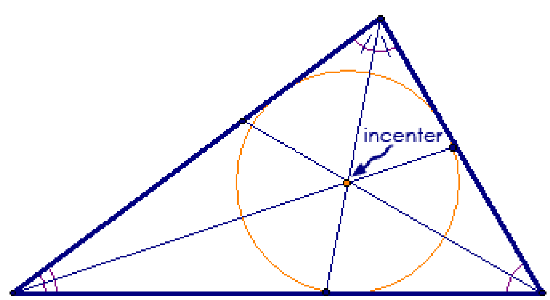
1.2.1 Centroid

A *median* of a triangle is constructed by connecting the midpoint of one side with the opposite vertex. The three medians of a triangle are concurrent and their point of intersection is called the *centroid*. It is a known fact that the centroid coordinates for a triangle with vertices $A = (A_x, A_y)$, $B = (B_x, B_y)$, $C = (C_x, C_y)$ is $(\frac{A_x+B_x+C_x}{3}, \frac{A_y+B_y+C_y}{3})$.



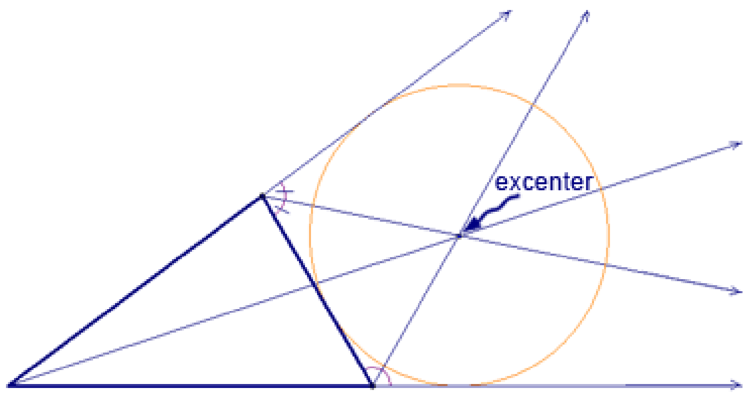
1.2.2 Incenter

The *incenter* of a triangle is formed by the intersection of the three angle bisectors of the vertices of the triangle. It is a known fact that the incenter coordinates for a triangle with vertices $A = (A_x, A_y)$, $B = (B_x, B_y)$, $C = (C_x, C_y)$ and corresponding side lengths a, b, c is $(\frac{A_x a+B_x b+C_x c}{a+b+c}, \frac{A_y a+B_y b+C_y c}{a+b+c})$.



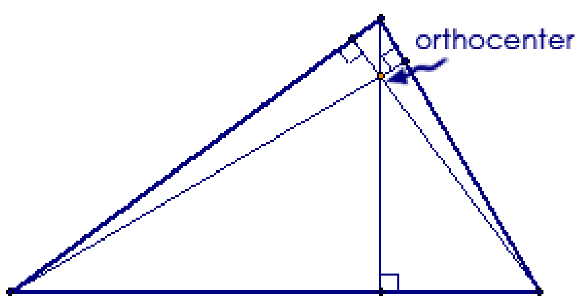
1.2.3 Excenter

An *excenter* of a triangle is formed by the intersection of the two exterior angle bisectors of the vertices of the triangle, and the interior angle bisector of the opposite side.



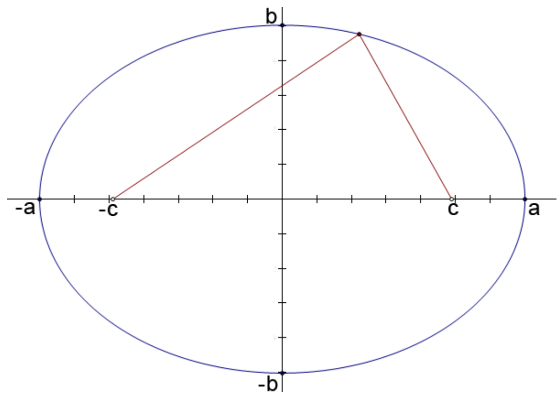
1.2.4 Orthocenter

An *altitude* of a triangle is a line that passes through the vertex and is perpendicular to the opposite side. The three altitudes of a triangle intersect at a single point called the *orthocenter*.



1.2.5 Ellipse

An ellipse is the set of all points the sum of whose distances from two fixed points (called the foci) is a positive integer, $2a$. Such an ellipse can be written as $\{\ (x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ \}$.



2 Theorems

2.1 Centroid

The centroid is an especially simple triangle center, found at the intersection of the three medians of a triangle, and a few minutes playing with Geometer's Sketchpad were enough to convince us that the centroid traces an ellipse, regardless of where the base of the triangle is situated. (This already marks a significant departure from the work done by the previous group, wherein the bases of the triangles considered always lay on the circle being traced.) Here, then, is a theorem which confirms these observations.

Theorem 2.1.1 *Let e_1 be an ellipse with semimajor and minor axes equal to a and b respectively and let $D = (x_1, y_1)$ and $E = (x_2, y_2)$ be two points in the plane. Then, if $F = (x_3, y_3)$ is any point on the ellipse e_1 the centroid of $\triangle DEF$ lies on a second ellipse, e_2, which is centered at $(\frac{x_1+x_2}{3}, \frac{y_1+y_2}{3})$ with semimajor and minor axes $\frac{a}{3}$ and $\frac{b}{3}$, respectively.*

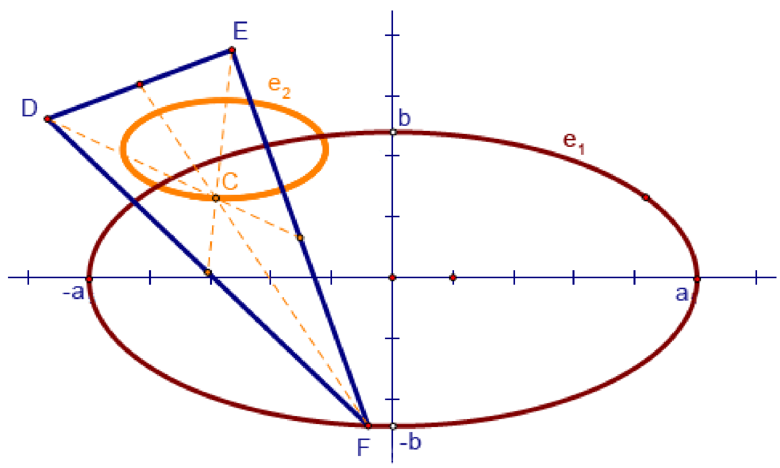


Figure 2

Proof Let e_1, D, E, and F be as stated. Then, without loss of generality, let $e_1 = \{\ (x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ \}$.

As D and E are any two points in the plane and F is on e_1, we can choose x_1, x_2, x_3, y_1, y_2, and y_3 such that $D = (x_1, y_1)$, $E = (x_2, y_2)$, $F = (x_3, y_3)$ and $\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} = 1$.

Let C be the centroid of $\triangle DEF$. Then we know $C = (x_4, y_4)$ such that $x_4 = \frac{x_1+x_2+x_3}{3}$, and $y_4 = \frac{y_1+y_2+y_3}{3}$. (See Figure 2)

Now, we wish to show that C lies on a second ellipse, e_2 centered at $(\frac{x_1+x_2}{3}, \frac{y_1+y_2}{3})$ and with semimajor and minor axes $\frac{a}{3}$ and $\frac{b}{3}$, respectively.

In order to show that C is on e_2 we simply need to show that

$$\frac{(x_4 - \frac{x_1+x_2}{3})^2}{(\frac{a}{3})^2} + \frac{(y_4 - \frac{y_1+y_2}{3})^2}{(\frac{b}{3})^2} = 1.$$

Thus let us consider the above equation, with respect to our new coordinates:

$$\frac{(x_4 - \frac{x_1+x_2}{3})^2}{(\frac{a}{3})^2} + \frac{(y_4 - \frac{y_1+y_2}{3})^2}{(\frac{b}{3})^2} = \frac{(\frac{x_1+x_2+x_3}{3} - \frac{x_1+x_2}{3})^2}{(\frac{a}{3})^2} + \frac{(\frac{y_1+y_2+y_3}{3} - \frac{y_1+y_2}{3})^2}{(\frac{b}{3})^2}$$

$$= \frac{(\frac{x_3}{3})^2}{(\frac{a}{3})^2} + \frac{(\frac{y_3}{3})^2}{(\frac{b}{3})^2} = \frac{\frac{x_3^2}{3^2}}{\frac{a^2}{3^2}} + \frac{\frac{y_3^2}{3^2}}{\frac{b^2}{3^2}} = \frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} = 1,$$

as (x_3, y_3) is on e_1.

Thus $C = (x_4, y_4)$ is on an ellipse e_2 described by

$$\frac{(x - \frac{x_1+x_2}{3})^2}{(\frac{a}{3})^2} + \frac{(y - \frac{y_1+y_2}{3})^2}{(\frac{b}{3})^2} = 1.$$

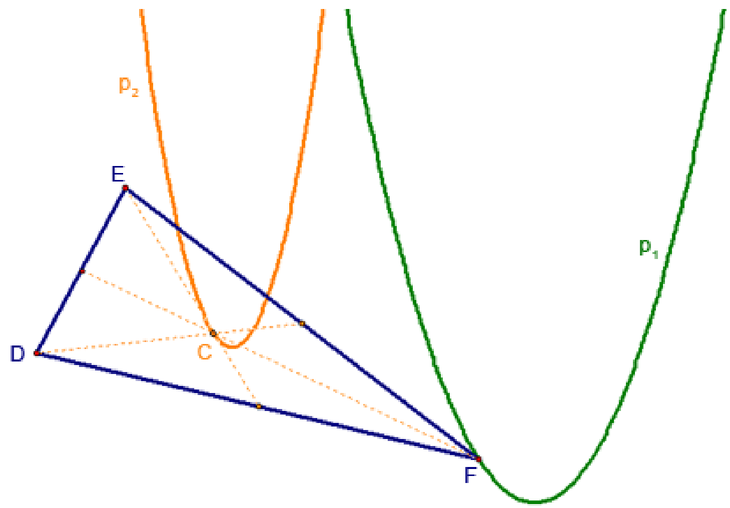
which is centered at $(\frac{x_1+x_2}{3}, \frac{y_1+y_2}{3})$ with semimajor and minor axes $\frac{a}{3}$ and $\frac{b}{3}$, respectively.

Using the same proof strategy as in Theorem 2.1.1, we can see that the centroid of a triangle traces hyperbolas and parabolas when one vertex of the triangle is tracing these conic sections.

Corollary 2.1.1 *Let h_1 be a hyperbola with semimajor and minor axes equal to a and b, respectively, and let $D = (x_1, y_1)$ and $E = (x_2, y_2)$ be two points in the plane. Then, if $F = (x_3, y_3)$ is any point on the hyperbola h_1 the centroid of $\triangle DEF$ lies on a second hyperbola, h_2, which is centered at $(\frac{x_1+x_2}{3}, \frac{y_1+y_2}{3})$ with semimajor and minor axes $\frac{a}{3}$ and $\frac{b}{3}$, respectively.*

Proof This proof follows immediately from that of the ellipse, the only alteration being a change from a + to a − throughout.

Corollary 2.1.2 *Let p_1 be a parabola with a equal to the distance from the vertex to both the focus and the directix and let $D = (x_1, y_1)$ and $E = (x_2, y_2)$ be two points in the plane. Then, if $F = (x_3, y_3)$ is any point on the parabola p_1, the centroid of $\triangle DEF$ lies on a second parabola, p_2, with vertex at $(\frac{x_1+x_2+x_3-x_3\sqrt{3}}{3}, \frac{y_1+y_2}{3})$ and the distance from the vertex to the focus and the directrix is $\frac{a}{3}$.*



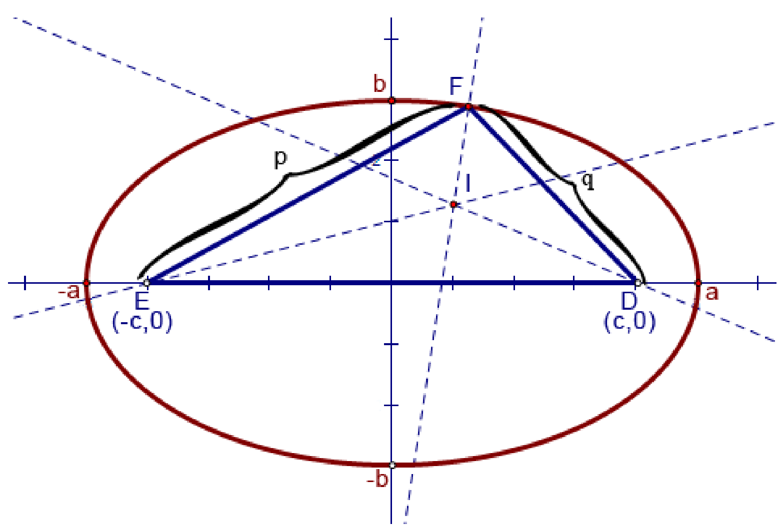
Proof Let p_1 and $\triangle DEF$ be as stated. Then we can write p_1 as $x^2 = 4ay$, and thus $x_3^2 = 4ay_3$. A little bit of formula manipulation similiar to that done in proof of theorem 2.1.1 reveals that the centroid C of $\triangle DEF$ lies on a curve p_2 described by the equation $(x - \frac{x_1+x_2+x_3-x_3\sqrt{3}}{3})^2 = 4\frac{a}{3}(y - \frac{y_1+y_2}{3})$ which is a parabola whose vertex is at $(\frac{x_1+x_2+x_3-x_3\sqrt{3}}{3}, \frac{y_1+y_2}{3})$ and whose distance from the vertex to the focus and the directrix is $\frac{a}{3}$.

2.2 Incenter

Having dealt with the centroid, the next step was to attempt to generalize the findings of the previous group. They had found that if the base of a triangle is placed on a circle, and the third point of that triangle is moved around that circle, the incenter would trace a circle. Unfortunately, a direct generalization of that result, substituting ellipse for circle, did not lead to any promising findings. The shapes generated were unrecognizable to us, and at any rate clearly not an ellipse. After much trial an error, it was eventually discovered that if the base of the triangle is placed on the foci of the ellipse, the incenter does appear to trace an ellipse. The following theorem confirms this result.

Theorem 2.2.1 *Let e_1 be an ellipse with semimajor and minor axes equal to a and b respectively and foci at $(c, 0)$ and $(-c, 0)$. Let $D = (c, 0)$ and $E = (-c, 0)$. Then, if F is any point on the ellipse e_1, the incenter of $\triangle DEF$ lies on a second ellipse described by the equation $\frac{x^2}{c^2} + \frac{y^2}{(\frac{bc}{a+c})^2} = 1$.*

Proof Let e_1 be an ellipse with semimajor and minor axes equal to a and b, respectively and foci at $(c, 0)$ and $(-c, 0)$. Let $\triangle DEF$ be formed by the points $D = (c, 0)$, and $E = (-c, 0)$, and $F = (x_1, y_1)$ such that F is on e_1. Let I be the incenter of $\triangle DEF$. Let p and q be the lengths of the two sides of the triangle as shown below:



So I has coordinates

$$(\frac{-cq+cp+2cx_1}{p+q+2c}, \frac{2cy_1}{p+q+2c})$$

$$= \frac{c}{2(a+c)}(p-q+2x_1, 2y_1).$$

Calculating $p-q$:

$$p = \sqrt{(x_1+c)^2+y_1^2} = \sqrt{(x_1+c)^2+b^2(1-\frac{x_1^2}{a^2})}$$

$$= \sqrt{(x_1+c)^2+(a^2-c^2)(1-\frac{x_1^2}{a^2})} = \frac{a^2+cx_1}{a}.$$

Similarly, $q = \frac{a^2-cx_1}{a}$. Thus, $p-q = \frac{2cx_1}{a}$. So,

$$I = \frac{c}{2(a+c)}(\frac{2cx_1}{a}+2x_1, 2y_1) = \frac{c}{2(a+c)}(\frac{2(c+a)x_1}{a}, 2y_1)$$

$$= (\frac{c}{a}x_1, \frac{c}{a+c}y_1).$$

Define a new ellipse, e_2, with equation $\frac{x^2}{c^2} + \frac{y^2}{(\frac{bc}{a+c})^2} = 1$. We want to show that I lies on e_2, regardless of where F lies on e_1. Thus, it suffices to show that

$$\frac{(\frac{c}{a}x_1)^2}{c^2} + \frac{(\frac{c}{a+c}y_1)^2}{(\frac{bc}{a+c})^2} = 1.$$

Considering the left hand side of this equality, we have:

$$\frac{(\frac{c}{a}x_1)^2}{c^2} + \frac{(\frac{c}{a+c}y_1)^2}{(\frac{bc}{a+c})^2} = \frac{\frac{c^2x_1^2}{a^2}}{c^2} + \frac{\frac{c^2y_1^2}{(a+c)^2}}{\frac{b^2c^2}{(a+c)^2}} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

And since (x_1, y_1) lies on e_1, we know $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, and therefore we get the desired result of

$$\frac{(\frac{c}{a}x_1)^2}{c^2} + \frac{(\frac{c}{a+c}y_1)^2}{(\frac{bc}{a+c})^2} = 1.$$

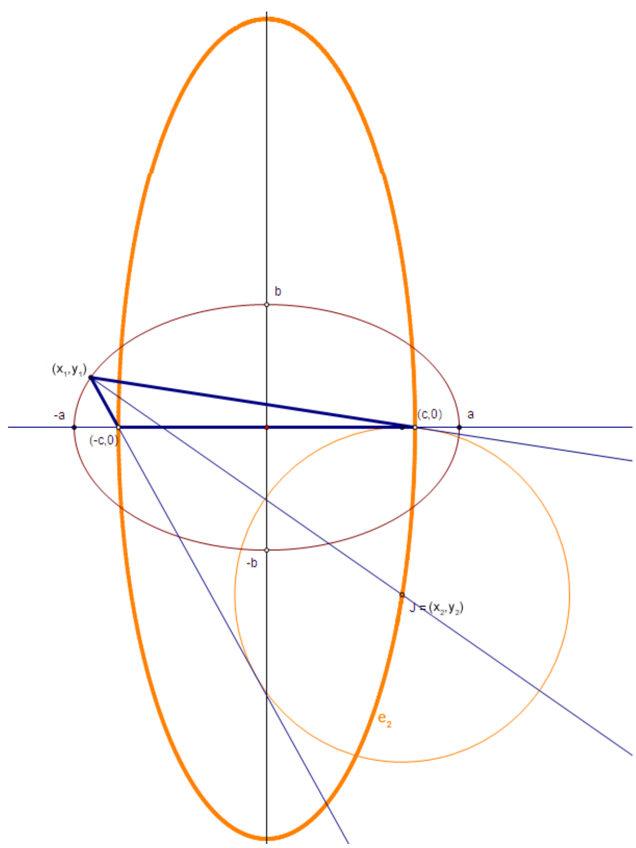
Thus I must lie on ellipse e_2.

2.3 Excenters

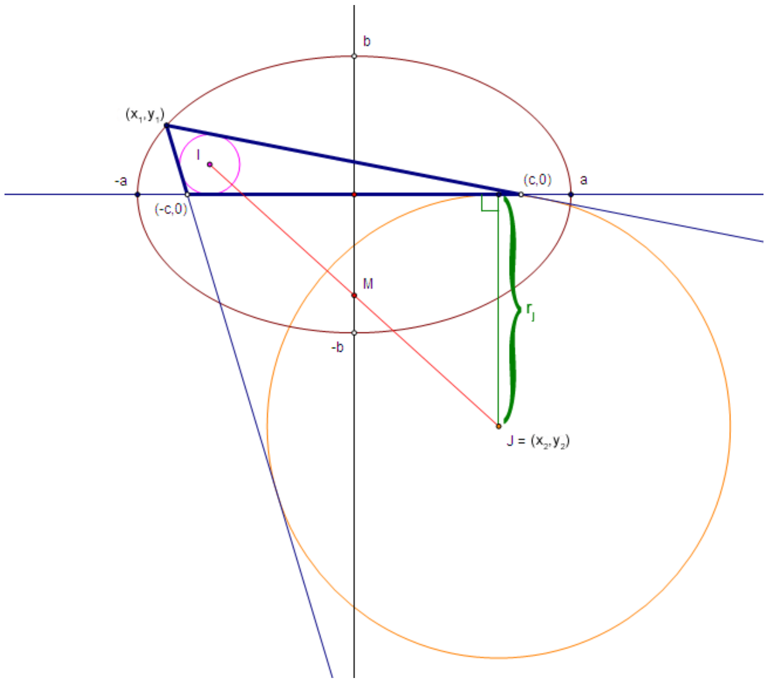
The excenters of a triangle are closely tied to the incenter, as both involve the bisection of the angles of the triangle. So it was not surprising to find that the excenters also traced recognizable paths when the triangles base was formed by the foci. However, different results were obtained depending on which excenter one considered. The excenter tangent to the base of the triangle traced an ellipse, while the other two excenters traced lines. The following two theorems demonstrate these results.

Theorem 2.3.1 *Given an ellipse e_1 and a triangle formed by the foci of the ellipse, $(-c, 0)$ and $(c, 0)$, and any third point on e_1, (x_1, y_1), the excenter of the base of the triangle lies on an ellipse*

$$e_2 : \frac{x^2}{c^2} + \frac{y^2}{(\frac{bc}{c-a})^2} = 1.$$



Proof Let e_1 be an ellipse with foci $(-c, 0)$ and $(c, 0)$ and axes a and b with $a > b$. Let $\triangle T$ be the triangle formed by $(-c, 0)$ and $(c, 0)$ and (x_1, y_1), where (x_1, y_1) lies on e_1. Let J be the excenter of the side of the triangle between the foci of e_2. Let I be the incenter of $\triangle T$, and let M be the midpoint of the segment IJ.



We begin by finding the coordinates (x_2, y_2) of J. The excircle center at J is tangent to the base of T and so a perpendicular line from J to the x-axis must have length r_J, the radius of the excircle. Thus $|y_2| = r_J$. Also, $r_J = \dfrac{\text{Area}(T)}{s - 2c}$, where $s = \dfrac{\text{Perimeter}(T)}{2}$. Thus,

$$r_J = \frac{\frac{1}{2}(2c)|y_1|}{\frac{1}{2}(2a + 2c) - 2c} = \frac{c|y_1|}{a - c}.$$

Thus $|y_2| = r_J = \frac{c|y_1|}{a-c}$. Since y_1 and y_2 are on different sides of the x-axis, $y_2 = \frac{cy_1}{c-a}$.

Now that we have y_2, we need x_2. We know that A is the center of the incenter-excenter circle, which passes through $(-c, 0)$ and $(c, 0)$. Thus A is equidistant between the two points and so must lie on the y-axis. And since M is the midpoint of segment $\overline{IJ}$, I and J are equidistant from the y-axis.

Thus $x_2 = \frac{-cx_1}{a}$ from our proof regarding the incenter. So $J = (\frac{-cx_1}{a}, \frac{cy_1}{c-a})$. Now we need only show that it lies on $e_2 : \frac{x^2}{c^2} + \frac{y^2}{(\frac{bc}{c-a})^2} = 1$. Substituting for x and y we get:

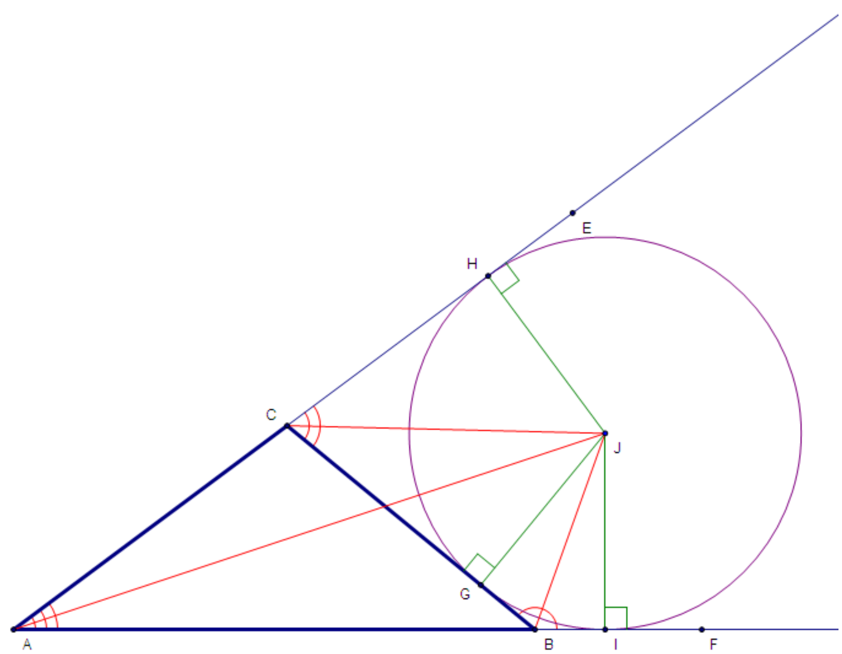
$$\frac{(\frac{-cx_1}{a})^2}{c^2} + \frac{(\frac{cy_1}{c-a})^2}{(\frac{bc}{c-a})^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

(as (x_1, y_1) lies on e_1).

Thus J lies on the ellipse e_2.

Theorem 2.3.2 *Let e_1 be an ellipse formed by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let $(-c, 0)$ and $(c, 0)$ be the foci of e_1. Let $\triangle T$ be a triangle formed by these two points and an arbitrary third point (x_1, y_1) that lies on e_1. Let J be the excenter of $\triangle T$ opposite $(-c, 0)$. Then J lies on the line $x = a$.*

> **Lemma.** Let $\triangle ABC$ be any triangle. Let J be the excenter opposite A, and let I be the point formed by dropping a perpendicular to $\overline{AB}$ from J. Then $\overline{AI}$=$\frac{1}{2}$Perimeter(ABC)



Let the set up be as above. Extend $\overline{AB}$ to E and $\overline{AC}$ to F. Bisect $\angle CAB$, $\angle FCB$, and $\angle CBE$. These meet at J. From J, drop perpendiculars to $\overline{BC}$, $\overline{AF}$, and $\overline{AE}$ at G, H, and I, respectively. G, H, and I are on the excircle centered at J and so $\overline{JG} \approx \overline{JH} \approx \overline{JI}$. By AAS, $\triangle HCJ \approx \triangle GCJ$, $\triangle GBJ \approx \triangle IBJ$, and $\triangle HAJ \approx \triangle GAJ$. From this we learn the following: $\overline{AH} \approx \overline{AI}$, $\overline{CH} \approx \overline{CG}$, and $\overline{BG} \approx \overline{BI}$. Now let's consider $2(\overline{AI})$:

$$2(\overline{AI}) = \overline{AI} + \overline{AH} = (\overline{AB} + \overline{BI}) + (\overline{AC} + \overline{CH})$$
$$= \overline{AB} + \overline{BG} + \overline{AC} + \overline{CG} = \overline{AB} + \overline{AC} + \overline{BC}$$

$$=\text{Perimeter}(\ ABC).$$

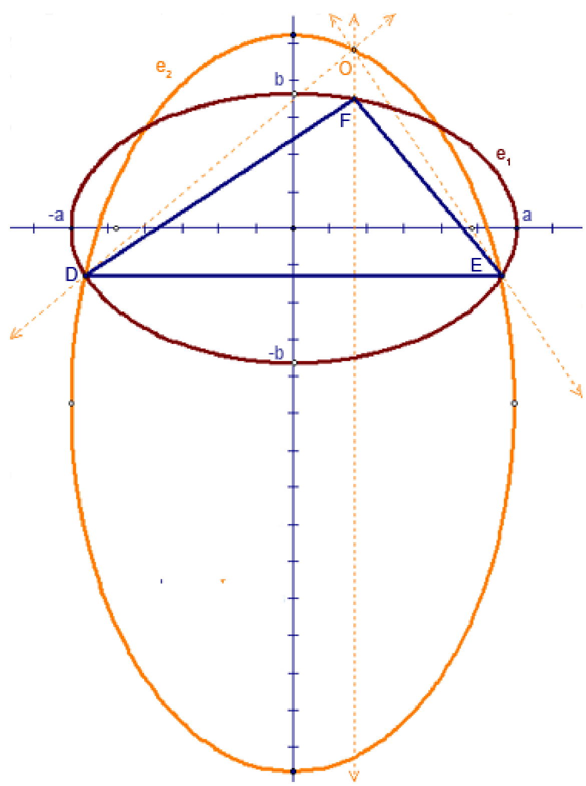
Therefore $\overline{AI}=\frac{1}{2}\text{Perimeter}(ABC)$, as desired.

Proof Consider the perpendicular dropped form J to the x-axis at I. By the above lemma, the distance from $(-c, 0)$ to I is equal to $\frac{1}{2}\text{Perimeter}(\triangle T)$. And since $\triangle T$ has two points on the foci and its third on e_1, its perimeter is constant and equal to $2(a + c)$. Thus the distance from $(-c, 0)$ to I is $a + c$, and so I lies at $(a, 0)$. Thus the x-coordinate of J is always a and so J lies on the line $x = a$.

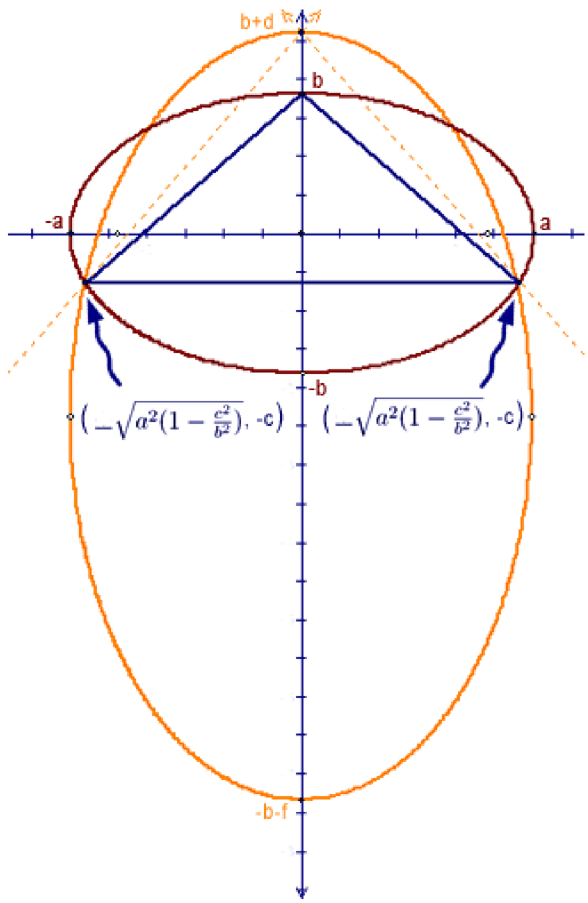
2.4 Orthocenter

The final triangle center to consider was the orthocenter. Here the direct generalization seemed valid: the orthocenter of a triangle whose three vertices lie on an ellipse traces an ellipse as one of the vertices moves around the ellipse. However, there was clearly more to it than that. Depending on where on the ellipse the base of the triangle is located, the orthocenters traced-ellipse changes in both size and position. In order to simplify the problem, we have first shown that if the base of the triangle is parallel to one of the axes, the orthocenter does in fact trace an ellipse. This theorem is proven below. We also include two conjectures, which, if proven, would prove the result for any triangle with all three points lying on an ellipse.

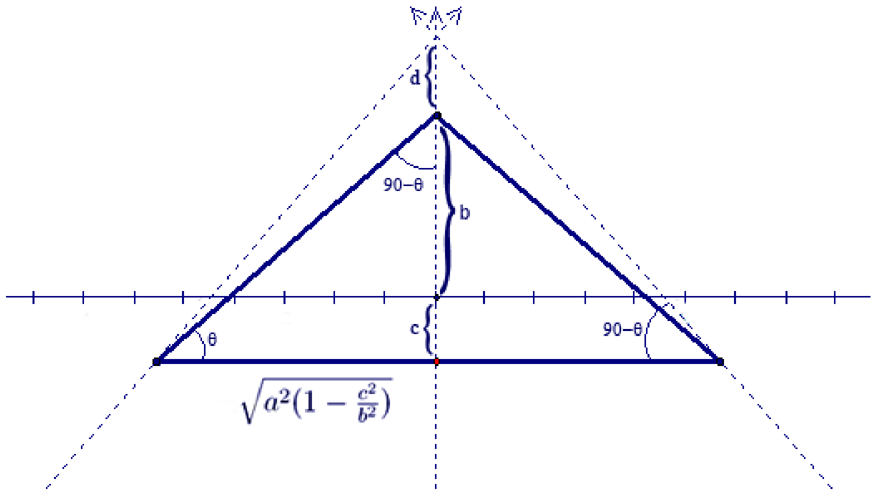
Theorem 2.4.1 *Let e_1 be an ellipse with semi-major axis along the x-axis such that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$. Inscribe a triangle in e_1 such that the base of the triangle is parallel to the x-axis. We wish to show that the orthocenter of such a triangle traces an ellipse, e_2, with semi-major axis along the y-axis, as its third vertex traverses e_1.*



Proof We begin by calculating the length of the semi-major and minor axes of e_2, such that the orthocenter of a triangle inscribed in the ellipse e_1 will trace e_2. First we notice that the vertices of the base of the triangle must be located at the points at which e_1 and e_2 meet. It is fairly straightforward to determine whether the center of e_2 lies above or below the center of e_1 based on the location of the base of the triangle. Since the base of our triangle lies below the semi-major axis of e_1 (see diagram), the center of e_2 will lie below that of e_1.



Let us call the y-coordinate of the intersection of e_1 and e_2 (and hence the base of our triangle) $-c$. Since the vertices of the base are on both e_1 and e_2 we can use our equation for e_1 to find that the x-coordinates of the vertices are $\pm\sqrt{a^2(1-\frac{c^2}{b^2})}$.



From the enlarged diagram (above)

$$\tan(90-\theta) = \frac{\sqrt{a^2(1-\frac{c^2}{b^2})}}{b+c},$$

and also

$$\tan(90-\theta) = \frac{(b+c+d)}{\sqrt{a^2(1-\frac{c^2}{b^2})}}.$$

Setting the two equal: $d = \frac{a^2(1-\frac{c^2}{b^2})-(b+c)^2}{b+c}$.

Similarly, we find $f = \frac{a^2(1-\frac{c^2}{b^2})-(b-c)^2}{b-c}$.
Since the major axis of e_2 is $d+b+c+(b-c)+f$, we have

$$d+2b+f = \frac{a^2(1-\frac{c^2}{b^2})(b+c)^2}{(b+c)} = \frac{2a^2}{b}.$$

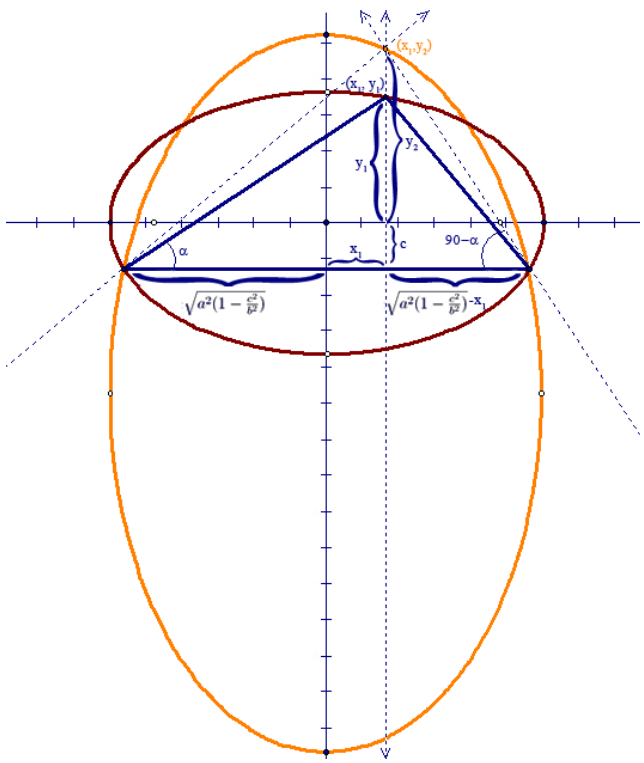
It is clear that the length of the semi-minor axis of e_2 must be a if it is to be traced from the orthocenter of e_1 (since the perpendicular dropped from the moving vertex will determine the x-range of the orthocenter's trace). Thus, we hope to show that the trace of the orthocenter of a triangle with base of height $-c$ inscribed in ellipse e_1, with formula $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, will trace ellipse e_2, with formula $\frac{x^2}{a^2}+\frac{(y-g)^2}{(\frac{a^2}{b})^2}=1$, as its third vertex moves along e_1.

First we wish to calculate g:

$$g = b + d - \frac{a^2}{b} = \frac{b(b+c) + a^2(1 - \frac{c^2}{b^2}) - (b+c)^2 - (\frac{a^2}{b})(b+c)}{b+c} = -c - \frac{a^2 c}{b^2}.$$

Thus we can rewrite the equation for e_2: $\frac{x^2}{a^2} + \frac{(y+(c+\frac{a^2 c}{b^2}))^2}{(\frac{a^2}{b})^2} = 1$.

Now we wish to show that for an arbitrary point (x_1,y_1) on e_2, this point is the orthocenter of the triangle we have inscribed in e_1. We know that the altitude from the moving vertex must be vertical (since the base of the triangle is horizontal), so let us call the location of the third vertex of the triangle (x_1, y_2).



From the diagram (above) we have

$$\tan(\alpha) = \frac{y_1 + c}{x_1 + \sqrt{a^2(1 - \frac{c^2}{b^2})}},$$

and

$$\tan(90-\alpha) = \frac{y_2 + c}{\sqrt{a^2(1-\frac{c^2}{b^2})} - x_1}$$

which implies

$$y_2 = \frac{\sqrt{a^2(1-\frac{c^2}{b^2})} - x_1}{\tan(\alpha)} - c.$$

Substituting, we get

$$y_2 = \frac{(\sqrt{a^2(1-\frac{c^2}{b^2})} - x_1)(\sqrt{a^2(1-\frac{c^2}{b^2})} + x_1)}{y_1 + c} - c$$

$$= \frac{-cy_1 - c^2 + a^2 - \frac{a^2c^2}{b^2} - x_1^2}{y_1 - c}$$

We wish to show

$$\frac{x_1^2}{a^2} + \frac{(y_2 + (c + \frac{a^2c}{b^2}))^2}{(\frac{a^2}{b})^2} = 1.$$

So

$$\frac{x_1^2}{a^2} + \frac{(\frac{-c(y_1+c)+a^2(1-\frac{c^2}{b^2})-x_1^2}{y_1+c} + (c + \frac{a^2c}{b^2}))^2}{(\frac{a^2}{b})^2}$$

$$= \frac{x_1^2}{a^2} + \frac{(\frac{-cy_1-c^2+a^2-\frac{a^2c^2}{b^2}-a^2+\frac{a^2y_1^2}{b^2}+cy_1+c^2+\frac{a^2cy_1}{b^2}+\frac{a^2c^2}{b^2}}{y_1+c})^2}{(\frac{a^2}{b})^2}$$

$$= \frac{x_1^2}{a^2} + (\frac{(\frac{b}{a^2})(\frac{a^2y_1^2}{b^2} + \frac{a^2cy_1}{b^2})}{y_1 + c})^2 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

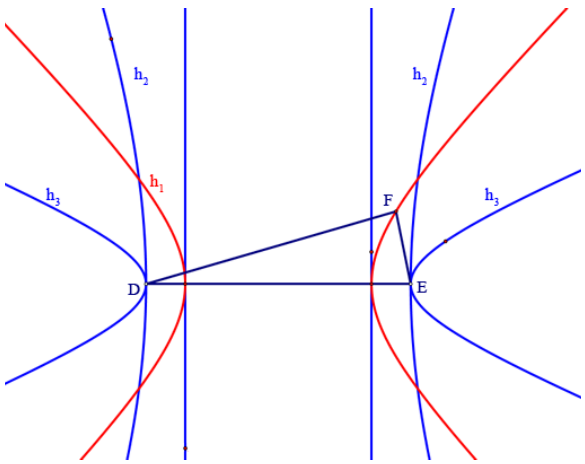
Thus, for any arbitrary point on e_2, this point is the orthocenter of a triangle inscribed in ellipse e_1, with the vertices of the base of the triangle located at the intersection of e_1 and e_2.

3 Conjectures

In our time investigating special points of triangles and conic sections we arrived at several conjectures that we have been unable to prove but believe to be true.

3.1 Incenters and Excenters

At the end of our investigation we decided also to investigate hyperbolas as well as ellipses, and we believe that the incenters and excenters of a triangle whose base is on the foci of a hyperbola trace out two hyperbolas and two parallel lines.



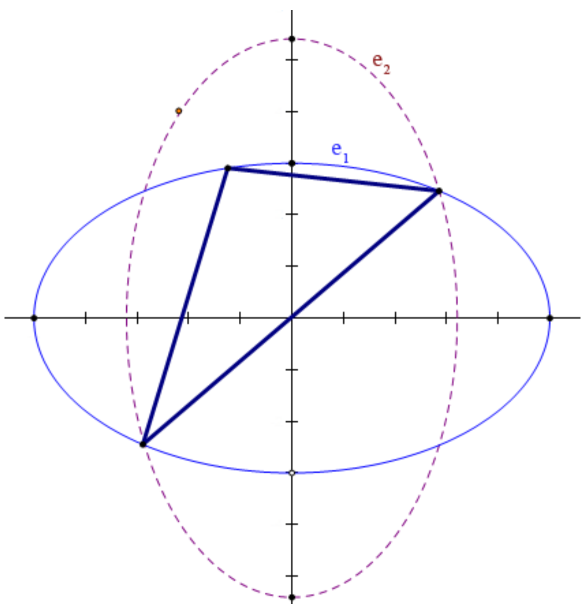
Conjecture 3.1.1 *Let h_1 be a hyperbola with foci D and E. Then if F is any point on the hyperbola h_1 the excenters of the sides $\overline{DF}$ and $\overline{EF}$ lie on one of two other hyperbolas h_2 and h_3 with vertices at D and E.*

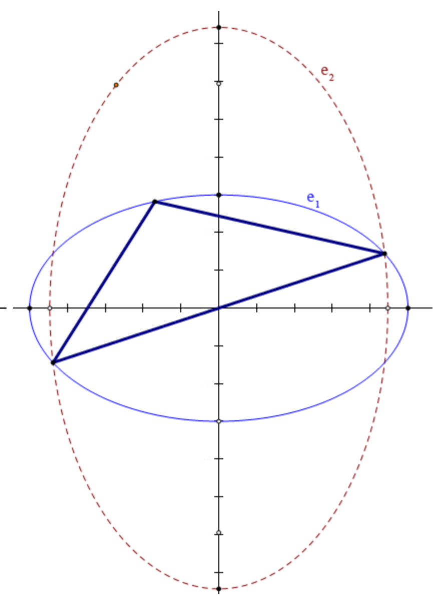
Conjecture 3.1.2 *Let h_1 be a hyperbola with foci D and E. Then if F is any point on the hyperbola h_1 the excenters of the side $\overline{DE}$ and the incenter of $\triangle DEF$ lie on one of the lines through the vertices of h_1 and perpendicular to $\overline{DE}$.*

3.2 Orthocenter

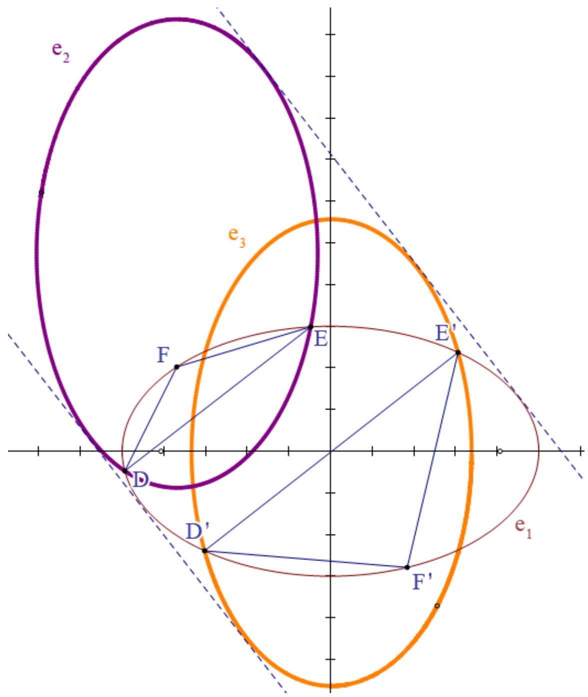
In this paper we proved a theorem that we believe to be a case of a larger theorem. We believe that the orthocenter of a triangle whose three vertices lie on an ellipse traces an ellipse as one of the vertices moves around the ellipse. However, we have only shown that the orthocenter of a triangle whose three vertices lie on an ellipse and whose base is parallel to the semi-major or minor axes traces out an ellipse. Here we have included two conjectures which, if proven, would show that the orthocenter of *any* triangle whose three vertices lie on an ellipse traces out an ellipse as one of the vertices moves around the ellipse.

Conjecture 3.2.1 *Let e_1 be an ellipse centered at the origin. Inscribe a triangle in e_1 such that the origin lies on the base of the triangle. We wish to show that the orthocenter of such a triangle traces an ellipse, e_2, centered at the origin with semi-major and minor axes determined by the angle of the base, as its third vertex traverses e_1.*



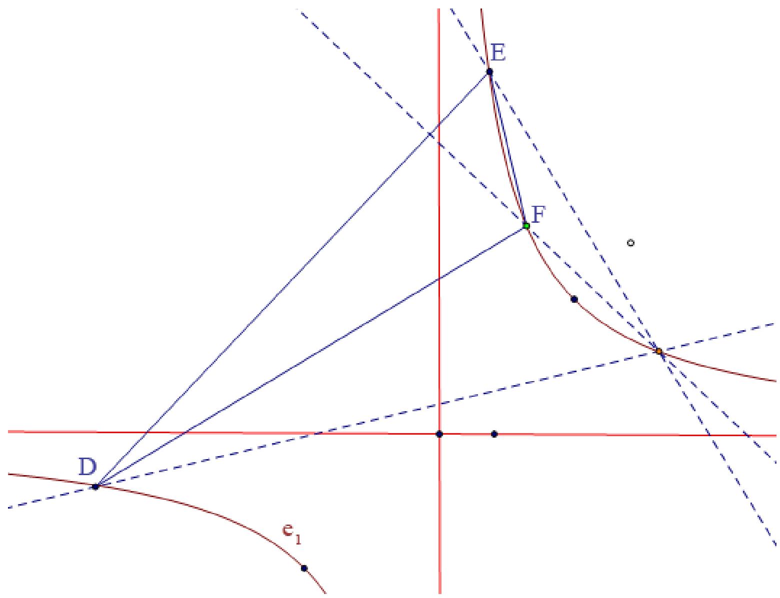


Conjecture 3.2.2 *Let e_1 be an ellipse centered at the origin. Inscribe a triangle DEF in e_1 such that the origin does not lie on $\overline{DE}$. We wish to show that the orthocenter of such a triangle traces an ellipse, e_2, whose center is determined by the distance of the base to the origin and whose semi-major and minor axes are the same as the ellipse e_3 traced by a triangle $D'E'F'$ whose base $\overline{D'E'}$ is parallel to $\overline{DE}$ and contains the origin, as its third vertex traverses e_1.*



At the end of our investigation we also examined the orthocenter of a triangle on a hyperbola. We believe that the orthocenter of any triangle whose vertices lie on a given hyperbola traces out that very hyperbola.

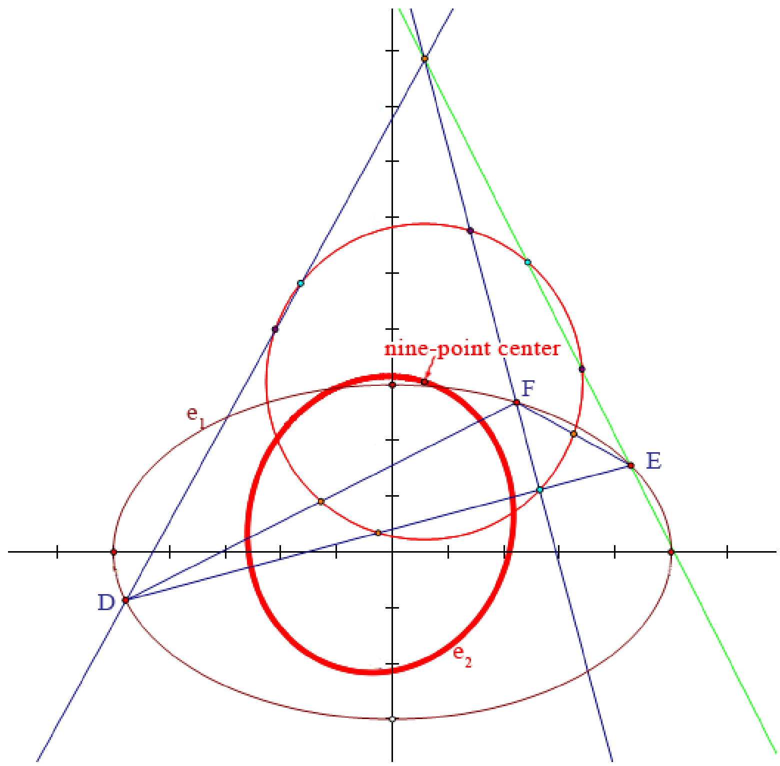
Conjecture 3.2.3 *Let h_1 be a hyperbola and $\triangle DEF$ be a triangle with all three vertices on h_1. Then the orthocenter of $\triangle DEF$ lies on h_1 as well.*



3.3 Nine-point center

We actually began our whole investigation with the nine-point center. The nine-point center is the center of the nine-point circle. This circle passes through the midpoints of the triangle's sides and the points where the altitudes intersect the triangle's sides.

Conjecture 3.3.1 *Let e_1 be an ellipse with $\triangle DEF$ inscribed within it. As F orbits around e_1, the nine-point center traces a second ellipse, e_2.*



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