INCOMPLETE BALANCING AND LUCAS-BALANCING NUMBERS

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The aim of this article is to establish some combinatorial expressions of balancing and Lucas-balancing numbers and investigate some of their properties.

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Key words: balancing numbers, Lucas-balancing numbers, incomplete balancing numbers, incomplete Lucas-balancing numbers.

1. INTRODUCTION

The terms balancing numbers and Lucas-balancing numbers are used to describe the series of numbers generated by the recursive formulas $B_n = 6B_{n-1} - B_{n-2}$; $B_0 = 0$, $B_1 = 1$ with $n \geq 2$ and $C_n = 6C_{n-1} - C_{n-2}$; $C_0 = 1$, $C_1 = 3$, with $n \geq 2$ respectively [1,10]. The roots $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$ for both these sequences, the respective Binet formulas are $B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$ and $C_n = \frac{\lambda_1 + \lambda_2}{2}$ [1,10]. Many interesting results of balancing numbers and their related sequences can be found in [5,10–12].

In [4], Filipponi established two interesting classes of integers namely, incomplete Fibonacci numbers and incomplete Lucas numbers which were obtained from some of the well-known combinatorial forms of Fibonacci and Lucas numbers. He has also studied some of the congruence properties for incomplete Lucas numbers in [4]. Filipponi dreamt a glimpse of possible generalizations of incomplete Fibonacci and Lucas numbers which were fulfilled by some authors later [2,7–9,13]. In this article we establish some combinatorial expressions for balancing and Lucas-balancing numbers and introduce incomplete balancing and incomplete Lucas-balancing numbers.

Balancing sequence has been generalized in many ways. One important generalization of balancing numbers is the k-balancing numbers introduced in [6,12]. k-balancing numbers are defined recursively by

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1} \quad n \geq 2,$$

with initials $B_{k,0} = 0$ and $B_{k,1} = 1$. In [12], Ray has introduced sequence of balancing polynomials $\{B_n(x)\}_{n=0}^{\infty}$that are natural extension of k-balancing numbers and is defined recursively by

$$B_n(x) = \begin{cases} 1, & \text{if } \; n = 0 \\ 6x, & \text{if } \; n = 1 \\ 6xB_{n-1}(x) - B_{n-2}(x), & \text{if } \; n > 1. \end{cases}$$

Further, he has also established the Binet formula as

$$B_n(x) = \frac{\lambda_1^n(x) - \lambda_2^n(x)}{2\triangle},$$

where

$$\lambda_1(x) = 3x + \triangle, \lambda_2(x) = 3x - \triangle \text{ and } \triangle = \sqrt{9x^2 - 1}.$$

The first few balancing polynomials are

$$B_2(x) = 6x, B_3(x) = 36x^2-1, B_4(x) = 216x^3-12x, B_5(x) = 1296x^4-108x^2+1.$$

The derivatives of balancing polynomials in the form of convolution of these polynomials are also presented in [12].

In a similar manner, the n^{th} Lucas-balancing polynomial $C_n(x)$ is defined as

$$C_n(x) = \begin{cases} 1, & \text{if } \; n = 0 \\ 3x, & \text{if } \; n = 1 \\ 6xC_{n-1}(x) - C_{n-2}(x), & \text{if } \; n > 1, \end{cases}$$

and its Binet formula is

$$C_n(x) = \frac{\lambda_1^n(x) + \lambda_2^n(x)}{2}.$$

The first few Lucas-balancing polynomials are

$$\begin{aligned} C_2(x) = 18x^2 - 1, C_3(x) = 108x^3 - 9x, C_4(x) = 648x^4 - 72x^2 + 1, C_5(x) \\ = 3888x^5 - 540x^3 + 15x. \end{aligned}$$

In this article, authors aim is to establish some combinatorial expressions of balancing and Lucas-balancing numbers and investigate some of their properties.

2. SOME COMBINATORIAL EXPRESSIONS OF BALANCING AND LUCAS BALANCING NUMBERS

In this section, we establish some combinatorial expressions for balancing and Lucas-balancing numbers. These expressions lead to form some combinatorial expressions for balancing and Lucas-balancing polynomials.

THEOREM 2.1. *Let B_n and C_n denote n^{th} balancing and Lucas-balancing numbers respectively, then*

$$B_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n-1-j}{j} 6^{n-2j-1}, \tag{2.1}$$

$$C_n = 3 \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} 6^{n-2j-1}, \tag{2.2}$$

where $\lfloor . \rfloor$ denotes the floor function.

Proof. From the Binet formulas for both balancing and Lucas-balancing numbers and by well-known Waring formulas [3],

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n-1-j}{j} (\lambda_1\lambda_2)^j (\lambda_1 + \lambda_2)^{n-2j-1},$$

$$2C_n = \lambda_1^n + \lambda_2^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (\lambda_1\lambda_2)^j (\lambda_1 + \lambda_2)^{n-2j},$$

and the result follows as $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = 6$. □

Indeed, the combinatorial expression for polynomial $B_n(x)$ leads to

$$B_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n-1-j}{j} (6x)^{n-1-2j} \quad \text{for} \quad n \geq 1,$$

which is shown in [12]. Similarly the combinatorial expression for polynomial $C_n(x)$ will be

$$C_n(x) = 3 \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} 6^{n-1-2j} x^{n-2j} \quad \text{for} \quad n \geq 1.$$

Further, their derivatives are respectively given by

$$B'_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j (n-1-2j) \binom{n-1-j}{j} 6^{n-1-2j} x^{n-2-2j} \text{ for } n \geq 1, \tag{2.3}$$

$$C'_n(x) = 3 \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n(n-2j)}{n-j} \binom{n-j}{j} (6x)^{n-1-2j} \quad \text{for} \quad n \geq 1. \tag{2.4}$$

Clearly, $B_0'(x) = C_0'(x) = 0$. Some simple properties of the polynomials $B_n'(x)$ and $C_n'(x)$ can be derived from the Binet formulas. In fact, letting

$$\lambda_1'(x) = \frac{d}{dx}(\lambda_1(x)) = \frac{3\lambda_1(x)}{\Delta},\ \lambda_2'(x) = \frac{d}{dx}(\lambda_2(x)) = \frac{-3\lambda_2(x)}{\Delta},$$

which follows

$$(\lambda_1^n(x))' = \frac{d}{dx}(\lambda_1^n(x)) = \frac{3n\lambda_1^n(x)}{\Delta},\ (\lambda_2^n(x))' = \frac{d}{dx}(\lambda_2^n(x)) = \frac{-3n\lambda_2^n(x)}{\Delta},$$

we can write

$$B_n'(x) = \frac{d}{dx}\left(\frac{\lambda_1^n(x) - \lambda_2^n(x)}{2\Delta}\right) = \frac{3nC_n(x) - 9xB_n(x)}{\Delta^2}, \tag{2.5}$$

$$C_n'(x) = \frac{d}{dx}(\lambda_1^n(x) + \lambda_2^n(x)) = 3nB_n(x). \tag{2.6}$$

3. INCOMPLETE BALANCING AND LUCAS-BALANCING NUMBERS

The combinatorial expressions (2.1) and (2.2) give rise to two interesting classes of integers $B_n(k)$ and $C_n(k)$, for integral values n and k. We call these integers as incomplete balancing numbers and incomplete Lucas-balancing numbers respectively which will be defined subsequently. In this section, authors aim to establish certain properties of incomplete balancing and incomplete Lucas-balancing numbers.

Definition 3.1. The incomplete balancing numbers $B_n(k)$ be defined as for any natural number n,

$$B_n(k) = \sum_{j=0}^{k}(-1)^j \binom{n-1-j}{j} 6^{n-2j-1}, \quad 0 \le k \le \tilde{n}, \tag{3.7}$$

where $\tilde{n} = \lfloor \frac{n-1}{2} \rfloor$.

The numbers $B_n(k)$ are shown in the Table 1 for the first few values of n and the corresponding admissible values of k.

Observation of Table 1 gives rise to,

$$B_n(0) = 6^{n-1} \quad \text{for} \quad n \ge 1,$$

$$B_n(\tilde{n}) = B_n \quad \text{for} \quad n \ge 1,$$

$$B_n(\tilde{n}-1) = \begin{cases} B_n \pm 3n, & \text{if } n \text{ is even} \\ B_n \pm 1, & \text{if } n \text{ is odd} \end{cases} \quad \text{for} \quad n \ge 3.$$

We define the incomplete Lucas-balancing numbers in the following manner:

TABLE 1

Incomplete balancing numbers

n/k	0	1	2	3	4	5	6
1	1						
2	6						
3	36	35					
4	216	204					
5	1296	1188	1189				
6	7776	6912	6930				
7	46656	40176	40392	40391			
8	279936	233280	235440	235416			
9	1679616	1353024	1372464	1372104	1372105		
10	10077696	7838208	8001504	7997184	7997214		
11	60466176	45349632	46656000	46610640	46611180	46611179	
12	362797056	262020096	272097792	271662336	271669896	271669860	
13	2176782336	1511654400	1587057120	1583318016	1583408736	1583407980	1583407981
14	13060694016	8707129344	9260322624	9227810304	9228790080	9228777984	9228778026
15	78364164096	50065993728	54050281344	53779804704	53789422464	53789259168	53789259420
16	470184984576	287335268351	315594558720	313418505600	313508724480	313506764928	313506778494

Definition 3.2. The incomplete Lucas-balancing numbers $C_n(k)$ be defined as for any natural number n,

$$C_n(k) = 3\sum_{j=0}^{k}(-1)^j \frac{n}{n-j}\binom{n-j}{j}6^{n-2j-1}, \quad 0 \leq k \leq \hat{n}, \tag{3.8}$$

where $\hat{n} = \left\lfloor \frac{n}{2} \right\rfloor$.

The numbers $C_n(k)$ are shown in Table 2 for the first few values of n and the corresponding admissible values of k.

TABLE 2

Incomplete Lucas-balancing numbers

n/k	0	1	2	3	4	5	6
1	3						
2	18	17					
3	108	99					
4	648	576	577				
5	3888	3348	3363				
6	23328	19440	19602	19601			
7	139968	112752	114264	114243			
8	2839808	653184	666144	665856	665857		
9	5038848	3779136	3884112	3880872	3880899		
10	30233088	21835008	22651488	22619088	22619538	22619537	
11	181398528	125971200	132129792	131830416	131836356	131836323	
12	1088391168	725594112	770943744	768331008	768399048	768398400	768398401
13	6530347008	4172166144	4499691264	4477856256	4478563872	4478554044	4478554083
14	39182082048	23944605696	26272553472	26096193792	26102984184	26102925216	26102926098
15	235092492288	137137287168	153463154688	152077471488	152140048848	152138987424	152139002544

Observation of Table 2 gives rise to,

$$C_n(0) = 3.6^{n-1} \quad \text{for} \quad n \geq 1,$$

$$C_n(\hat{n}) = C_n \quad \text{for} \quad n \geq 1,$$

$$C_n(\hat{n}-1) = \begin{cases} C_n \pm 1, & \text{if } n \text{ is even} \\ C_n \pm n, & \text{if } n \text{ is odd} \end{cases} \quad \text{for} \quad n \geq 3.$$

3.1. SOME IDENTITIES CONCERNING THE INCOMPLETE BALANCING NUMBERS $B_n(k)$

Like balancing and Lucas-balancing numbers, incomplete balancing numbers also satisfy the second order recurrence relation. The following result demonstrates this fact.

PROPOSITION 3.3. *The numbers $B_n(k)$ obey the second order recurrence relation*

$$B_{n+2}(k+1) = 6B_{n+1}(k+1) - B_n(k) \quad for \quad 0 \leq k \leq \lfloor (n-2)/2 \rfloor. \tag{3.9}$$

Proof. By virtue of the Definition 3.7,

$$\begin{aligned} 6B_{n+1}(k+1) - B_n(k) &= 6\sum_{j=0}^{k+1}(-1)^j \binom{n-j}{j} 6^{n-2j} - \sum_{j=0}^{k}(-1)^j \binom{n-1-j}{j} 6^{n-1-2j} \\ &= 6\sum_{j=0}^{k+1}(-1)^j \left[\binom{n-j}{j} + \binom{n-j}{j-1} + \binom{n}{-1} 6^n \right] 6^{n-2j} \\ &= \sum_{j=0}^{k+1}(-1)^j \binom{n-j+1}{j} 6^{n+1-2j}, \end{aligned}$$

and the result follows. □

Observe that, the relation (3.9) can be transformed into the non-homogeneous recurrence relation

$$B_{n+2}(k) = 6B_{n+1}(k) - B_n(k) - (-1)^{k+1} 6^{n-1-2k} \binom{n-1-k}{k}. \tag{3.10}$$

The relation (3.10) can be generalized as follows.

PROPOSITION 3.4. *Let $0 \leq k \leq \frac{n-h-1}{2}$. Then the identity*

$$\sum_{j=0}^{h}(-1)^{j+1} 6^j \binom{h}{j} B_{n+j}(k+j) = (-1)^{h+1} B_{n+2h}(k+h) \tag{3.11}$$

holds.

Proof. Using induction on h, the result (3.11) clearly holds for $h = 0$ and $h = 1$. Assume that the result holds for some $h > 1$. For the inductive step, we have

$$\sum_{j=0}^{h+1}(-1)^{j+1} 6^j \binom{h+1}{j} B_{n+j}(k+j)$$

$$\begin{aligned}
&= \sum_{j=0}^{h+1} (-1)^{j+1} 6^j \left[\binom{h}{j} + \binom{h}{j-1} \right] B_{n+j}(k+j) \\
&= (-1)^{h-1} B_{n+2h}(k+h) + \sum_{i=-1}^{h+1} (-1)^{i+2} 6^{i+1} \binom{h}{i} B_{i+n+1}(k+1+i) \\
&= (-1)^{h-1} B_{n+2h}(k+h) + \sum_{i=0}^{h} (-1)^{i+2} 6^{i+1} \binom{h}{i} B_{i+n+1}(k+1+i) \\
&= (-1)^{h-1} B_{n+2h}(k+h) - 6 \sum_{i=0}^{h} (-1)^{i+1} 6^{i} \binom{h}{i} B_{i+n+1}(k+1+i) \\
&= (-1)^{h-1} B_{n+2h}(k+h) - 6(-1)^{h-1} B_{n+2h+1}(k+1+h) \\
&= (-1)^{h-1} [B_{n+2h}(k+h) - 6B_{n+2h+1}(k+1+h)] \\
&= (-1)^{h} B_{n+2(h+1)}(k+1+h),
\end{aligned}$$

which completes the proof. $\square$

Using induction on m, the following identity can be proved analogously.

$$\sum_{j=0}^{h-m} (-1)^{j+1} 6^{j+m} \binom{h}{m+j} B_{n+j}(j) = (-1)^{h+1} B_{n+2h-m}(h-m), \tag{3.12}$$

where $h \geq m$ and $n \geq h-m+1$.

Notice that, the identity obtained from (3.12) by setting $m = 0$ is identical to the identity obtained from (3.11) for $k = 0$.

The following is an interesting relation concerning incomplete balancing numbers. The sum of all elements of the n^{th} row of the array of Table 1 is expressed in terms of balancing and Lucas-balancing numbers.

Proposition 3.5. *For incomplete balancing numbers $B_n(k)$, the following identity holds.*

$$\sum_{k=0}^{\tilde{n}} B_n(k) = \begin{cases} (3nC_n - B_n)/16, & \text{if } n \text{ is even} \\ (3nC_n + 7B_n)/16 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Recall that $\tilde{n} = \lfloor \frac{n-1}{2} \rfloor$. Then,

$B_n(0) + B_n(1) + \cdots + B_n(\tilde{n})$

$$\begin{aligned}
&= \binom{n-1-0}{0} 6^{n-1-0} + \left[(-1)^0 \binom{n-1-0}{0} 6^{n-1-0} + (-1)^1 \binom{n-1-1}{1} 6^{n-1-2} \right] \\
&\quad + \cdots + \left[(-1)^0 \binom{n-1-0}{0} 6^{n-1-0} + \cdots + (-1)^{\tilde{n}} \binom{n-1-\tilde{n}}{\tilde{n}} 6^{n-1-2\tilde{n}} \right]
\end{aligned}$$

$$= (\tilde{n}+1)\binom{n-1-0}{0}6^{n-1-0} + (\tilde{n}+1-1)(-1)^1\binom{n-1-1}{1}6^{n-1-2}$$
$$+\cdots+(-1)^{\tilde{n}}\binom{n-1-\tilde{n}}{\tilde{n}}6^{n-1-2\tilde{n}}$$
$$= \sum_{j=0}^{\tilde{n}}(\tilde{n}+1-j)(-1)^j\binom{n-1-j}{j}6^{n-1-2j}$$
$$= (\tilde{n}+1)\sum_{j=0}^{\tilde{n}}(-1)^j\binom{n-1-j}{j}6^{n-1-2j} - \sum_{j=0}^{\tilde{n}}(-1)^j j\binom{n-1-j}{j}6^{n-1-2j}$$
$$= (\tilde{n}+1)B_n - \sum_{j=0}^{\tilde{n}}(-1)^j j\binom{n-1-j}{j}6^{n-1-2j}.$$

The second term of the right hand side expression of the above equation leads to $\frac{B_n(8n+1)-3nC_n}{16}$ for $x=1$ in (2.3) and (2.5). For n is even, $\sum_{k=0}^{\tilde{n}} B_n(k)$ is $\frac{nB_n}{2} - \frac{B_n(8n+1)-3nC_n}{16}$ and for n is odd, $\sum_{k=0}^{\tilde{n}} B_n(k)$ is $\frac{(n+1)B_n}{2} - \frac{B_n(8n+1)-3nC_n}{16}$. This completes the proof. □

In [4], Filipponi has shown that for $n = 2^m$, where m is a non-negative integer, the incomplete Fibonacci numbers $F_n(k)$ is odd for all admissible values of k. The following result shows that for $n = 2m$, the incomplete balancing numbers $B_n(k)$ is even for all admissible values of k.

Proposition 3.6. *If $n = 2m$ for any non-negative integer m, then $B_n(k)$ is even for all admissible values of k.*

Proof. We prove this result by induction on m. The basic step is true for $m = 1$. In the inductive step, assume that the result is valid for all $m \le n$. Now, using the (3.9); we have

$$B_{2m+2}(k+1) = 6B_{2m+1}(k+1) - B_{2m}(k).$$

Clearly the first term is divisible by 2 and by the hypothesis the second term is also even and hence the result follows. □

Filipponi has also shown that for any prime p, the incomplete Lucas numbers satisfy the identity $L_p(k) \equiv 1 \ (mod\ p)$ for all admissible values of k in [4]. Whereas no such type of identity exists in incomplete Fibonacci numbers. To demonstrate this fact, consider the following example.

Example 3.7. Let $p = 7$. Consider $k = 1, 2, 3$ and observe that $F_7(1) = 6 \equiv -1 \ (mod\ 7)$, $F_7(2) = 12 \equiv -2 \ (mod\ 7)$ and $F_7(3) = 13 \equiv -1 \ (mod\ 7)$, respectively.

However, the similar type of identities do exist in both incomplete balancing and Lucas-balancing numbers. The following result demonstrates this fact.

PROPOSITION 3.8. *If $n = 2^m$ for any non-negative integer m, then for all admissible values of k, $B_n(k) \equiv 0 \ (mod\ \ n)$.*

Proof. By virtue of Definition (3.7),

$$B_n(k) = 6^{2^m-1} + \sum_{j=1}^{k} (-1)^j \binom{2^m - 1 - j}{j} 6^{2^m-1-2j}. \tag{3.13}$$

The first term on the right hand side (3.13) is an integer when $B_{2^m}(k)$ is divisible by 2^m for all $m \geq 0$. The second term is also an integer due to Filipponi [3], and thus the congruence follows. □

3.2. SOME IDENTITIES INVOLVING THE INCOMPLETE LUCAS-BALANCING NUMBERS $C_n(k)$

Balancing numbers and Lucas-balancing numbers are related by an identity $B_{n+1} - B_{n-1} = 2C_n$. Similar properties are valid for incomplete balancing and Lucas-balancing numbers.

PROPOSITION 3.9. *The identity*

$$B_{n+1}(k) - B_{n-1}(k-1) = 2C_n(k), \tag{3.14}$$

holds for $0 \leq k \leq \hat{n}$, where $\hat{n} = \lfloor \frac{n}{2} \rfloor$.

Proof. By virtue of Definition 3.7,

$$\begin{aligned} B_{n+1}(k) - B_{n-1}(k-1) &= \sum_{j=0}^{k} (-1)^j \binom{n-j}{j} 6^{n-2j} - \sum_{j=0}^{k-1} (-1)^j \binom{n-2-j}{j} 6^{n-2j-2} \\ &= \sum_{j=0}^{k} (-1)^j \binom{n-j}{j} 6^{n-2j} - \sum_{j=1}^{k} (-1)^{j-1} \binom{n-1-j}{j-1} 6^{n-2j} \\ &= \sum_{j=0}^{k} (-1)^j \left(\binom{n-1-j}{j-1} + \binom{n-j}{j} \right) 6^{n-2j} \end{aligned}$$

$$= 2\left[3\sum_{j=0}^{k}(-1)^j \frac{n}{n-j}\binom{n-1-j}{j-1}6^{n-2j-1}\right],$$

which ends the proof. □

Proposition 3.10. *The numbers $C_n(k)$ obey the second order recurrence relation*

$$C_{n+2}(k+1) = 6C_{n+1}(k+1) - C_n(k). \tag{3.15}$$

Proof. Using (3.14) and (3.9),

$$\begin{aligned} C_{n+2}(k+1) &= \frac{1}{2}\{B_{n+3}(k+1) - B_{n+1}(k)\} \\ &= \frac{1}{2}[\{6B_{n+2}(k+1) - B_{n+1}(k)\} - \{6B_n(k) - B_{n-1}(k-1)\}] \\ &= 3(B_{n+2}(k+1) - B_n(k)) + \frac{1}{2}(B_{n-1}(k-1) - B_{n+1}(k)) \\ &= 6C_{n+1}(k+1) - C_n(k), \end{aligned}$$

and the result follows. □

Notice that, the relation (3.15) can be transformed into the non-homogeneous recurrence relation

(3.16)

$$C_{n+2}(k) = 6C_{n+1}(k) - C_n(k) + (-1)^{k+2}\frac{3n}{n-k}\binom{n-k}{k}6^{n-2k-1} \quad \text{for} \quad n \geq 2k.$$

Proposition 3.11. *For $0 \leqslant k \leqslant \tilde{n}$, then the following identity holds.*

$$12\ C_n(k) = B_{n+2}(k) - B_{n-2}(k-2)\ . \tag{3.17}$$

Proof. In view of (3.14) and (3.15),

$$\begin{aligned} B_{n+2}(k) &= 2C_{n+1}(k) + B_n(k-1), \\ B_{n-2}(k-2) &= B_n(k-1) - 2C_{n-1}(k-1), \end{aligned}$$

from which the result follows. □

Using the relations (3.11) and (3.14), the identity (3.15) can be generalized as follows.

Proposition 3.12. *For $0 \leq k \leq \frac{n-h-1}{2}$, then*

$$\sum_{j=0}^{h}(-1)^{j+1}6^j\binom{h}{j}C_{n+j}(k+j) = (-1)^{h+1}C_{n+2h}(k+h)$$

holds.

The following is an interesting relation concerning incomplete Lucas-balancing numbers. The sum of all elements of the n^{th} row of the array of Table 2 is expressed in terms of balancing and Lucas-balancing numbers.

Proposition 3.13. *For incomplete Lucas-balancing numbers $C_n(k)$,*

$$\sum_{k=0}^{\hat{n}} C_n(k) = \begin{cases} C_n + n(3B_n - 2C_n)/2\ , & \text{if } n \text{ is even} \\ \{C_n + n(3B_n - 2C_n)\}/2\ , & \text{if } n \text{ is odd}\end{cases}.$$

Proof. Recall that $\hat{n} = \lfloor \frac{n}{2} \rfloor$. Therefore, we have

$$C_n(0) + C_n(1) + \cdots + C_n(\hat{n}) = (\hat{n}+1)C_n - 3\sum_{j=0}^{\hat{n}} (-1)^j j \frac{n}{n-j} \binom{n-j}{j} 6^{n-1-2j}.$$

Putting $x = 1$ in (2.4) and (2.6) and adopting the same procedure described earlier, we obtain

$$B_n = \sum_{j=0}^{\hat{n}} (-1)^j \frac{n-2j}{n-j} \binom{n-j}{j} 6^{n-1-2j}.$$

Therefore,

$$3\sum_{j=0}^{\hat{n}} (-1)^j j \frac{n}{n-j} \binom{n-j}{j} 6^{n-1-2j} = 3n(C_n - B_n)/2.$$

For n is even and odd, $\sum_{k=0}^{\hat{n}} C_n(k)$ is $(\frac{n}{2}+1)C_n - 3n(C_n - B_n)/2$ and $(\frac{n+1}{2})C_n - 3n(C_n - B_n)/2$ respectively. Therefore, the proof is completed. $\square$

Proposition 3.14. *If $n = 3^m$ for any non-negative integer m, then for all admissible values of k, $C_n(k) \equiv 0 \ (mod\ n)$.*

Proof. By virtue of Definition 3.16,

$$C_n(k) = 2^{3m-1}3^{3m} + 3^{m+1}\sum_{j=1}^{k} (-1)^j \frac{1}{3^m - j} \binom{3^m - j}{j} 6^{3^m-2j-1}. \tag{3.18}$$

The first term on the right hand side of (3.18) is an integer when $C_{3^m}(k)$ is divisible by 3^m for all $m \geq 0$. The second term is also an integer again due to Filipponi [4] and thus the congruence follows. $\square$

4. GENERATING FUNCTIONS OF THE INCOMPLETE BALANCING AND LUCAS-BALANCING NUMBERS

The generating functions for balancing and Lucas-balancing numbers play a vital role to find out many important identities for these numbers. In this section, we derive generating functions for both incomplete balancing and Lucas-balancing numbers.

The following result [7] is useful to prove the subsequent theorem.

LEMMA 4.1. *Let $\{s_n\}_{n\in\mathbb{N}}$ be a complex sequence satisfying the following non-homogeneous second-order recurrence relation:*

$$s_n = as_{n-1} + bs_{n-2} + r_n, \quad n > 1,$$

where $a, b \in \mathbb{C}$ and $r_n : \mathbb{N} \longrightarrow \mathbb{C}$ is a given sequence. Then the generating function $U(t)$ of s_n is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2},$$

where $G(t)$ denotes generating function of r_n.

The following are the generating functions for incomplete balancing and incomplete Lucas-balancing numbers.

THEOREM 4.2. *Let k be a fixed positive integer. Then*

$$\sum_{j=0}^{\infty} B_k(j)t^j = t^{2k+1}\frac{(1-6t)^{k+1}(B_{2k+1} - B_{2k}t) - (-1)^{k+1}t^2}{(1-6t+t^2)(1-6t)^{k+1}}, \tag{4.19}$$

$$\sum_{j=0}^{\infty} C_k(j)t^j = t^{2k}\frac{(1-6t)^{k+1}(C_{2k} - C_{2k-1}t) - (-1)^{k+1}t^2}{(1-6t+t^2)(1-6t)^{k+1}}. \tag{4.20}$$

Proof. By virtue of the identity (3.7), for $0 \le n < 2k+1$, $B_n(k) = 0$ and for other values of n and $r \ge 0$ integers, then $B_{2k+1+r}(k) = B_{2k+r}$. It follows from (3.10) that, for $n \ge 2k+3$,

$$B_n(k) = 6B_{n-1}(k) - B_{n-2}(k) - (-1)^{k+1}\binom{n-3-k}{n-3-2k}6^{n-3-2k}.$$

Letting $s_0 = B_{2k+1}(k), s_1 = B_{2k+2}(k), \ldots, s_n = B_{n+2k+1}$ and suppose that $r_0 = r_1 = 0$ and $r_n = (-1)^{k+1}6^{n-2}\binom{n-2+k}{n-2}$. It is easy to deduce the generating function of the sequence $\{r_n\}$ as $G(t) = \frac{(-1)^{k+1}t^2}{(1-6t)^{k+1}}$. Therefore by Lemma 4.1, the generating function of the incomplete balancing numbers satisfies the following:

$$U_k(t)(1-6t+t^2) + \frac{(-1)^{k+1}t^2}{(1-6t)^{k+1}} = B_{2k+1} + (B_{2k+2} - 6B_{2k+1})t,$$

where $U_k(t)$ is the generating function of the sequence $\{s_n\}$. It follows that $\sum_{j=0}^{\infty} B_k(j)t^j = t^{2k+1}U_k(t)$. In the proof of the identity (4.20), we use the following facts:
For $0 \leq n < 2k$, $C_n(k) = 0$, and for other values of n and $r \geq 0$ integers then, $C_{2k+1+r}(k) = C_{2k+r}$. It follows from (3.16) that, for $n \geq 2k+2$,

$$C_n(k) = 6C_{n-1}(k) - C_{n-2}(k) + (-1)^{k+2}\frac{3(n-2)}{(n-2-k)}\binom{n-2-k}{n-2-2k}6^{n-2k-3}.$$

Letting $s_0 = C_{2k}(k), s_1 = C_{2k+1}(k), \ldots, s_n = C_{n+2k}$ and suppose that $r_0 = r_1 = 0$ and

$$r_n = 3.6^{n-3}(-1)^{k+2}\frac{n-2+2k}{n-2+k}\binom{n-2+k}{n-2}.$$

The generating function of the sequence $\{r_n\}$ is $G_1(t) = \frac{(-1)^{k+2}3t^2(2-t)}{(1-6t)^{k+1}}$. Again from Lemma 4.1, the generating function of the incomplete Lucas-balancing numbers satisfies the equation:

$$U_k(t)(1-6t+t^2) + \frac{(-1)^{k+2}3t^2(2-t)}{(1-6t)^{k+1}} = C_{2k} + (C_{2k+1} - 6C_{2k})t.$$

Finally, we conclude that $\sum_{j=0}^{\infty} C_k(j)t^j = t^{2k}U_k(t)$. $\square$

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