Graphs and transformations, Mixed Exercise 4

1 a $y = x^2(x - 2)$

$0 = x^2(x - 2)$

So $x = 0$ or $x = 2$

The curve crosses the x-axis at (2, 0) and touches it at (0, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

$y = 2x - x^2$

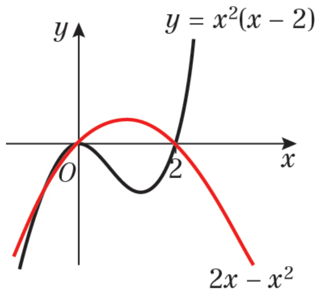
$= x(2 - x)$

As $a = -1$ is negative, the graph has a $\cap$ shape and a maximum point.

$0 = x(2 - x)$

So $x = 0$ or $x = 2$

The curve crosses the x-axis at (0, 0) and (2, 0).



b $x^2(x - 2) = x(2 - x)$

$x^2(x - 2) - x(2 - x) = 0$

$x^2(x - 2) + x(x - 2) = 0$

$x(x - 2)(x + 1) = 0$

So $x = 0$, $x = 2$ or $x = -1$

Using $y = x(2 - x)$:

when $x = 0$, $y = 0 \times 2 = 0$

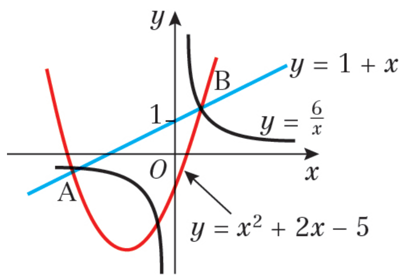
when $x = 2$, $y = 2 \times 0 = 0$

when $x = 0$, $y = (-1) \times 3 = -3$

The points of intersection are (0, 0), (2, 0) and (−1, −3).

2 a $y = \frac{6}{x}$ is like $y = \frac{1}{x}$.

$y = 1 + x$ is a straight line.



2 b $\frac{6}{x} = 1 + x$

$6 = x + x^2$

$0 = x^2 + x - 6$

$0 = (x + 3)(x - 2)$

So $x = 2$ or $x = -3$

Using $y = 1 + x$:

when $x = 2$, $y = 1 + 2 = 3$

when $x = -3$, $y = 1 - 3 = -2$

So A is (−3, −2) and B is (2, 3).

c Substituting the points A and B into $y = x^2 + px + q$:

A: $-2 = 9 - 3p + q$ **(1)**

B: $3 = 4 + 2p + q$ **(2)**

(1) − **(2)**:

$-5 = 5 - 5p$

$p = 2$

Substituting in **(1)**:

$-2 = 9 - 6 + q$

$q = -5$

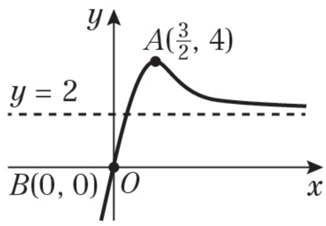
d $y = x^2 + 2x - 5$

As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

$y = (x + 1)^2 - 6$

So the minimum is at (−1, −6).

3 a f(2x) is a stretch with scale factor $\frac{1}{2}$ in the x-direction.

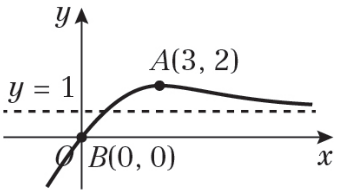


$A'\left(\frac{3}{2}, 4\right)$, $B'(0, 0)$

The asymptote is $y = 2$.

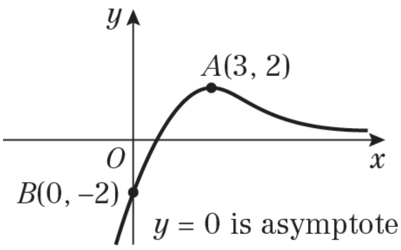
b $\frac{1}{2}$f(x) is a stretch with scale factor $\frac{1}{2}$ in the y-direction.

3 b



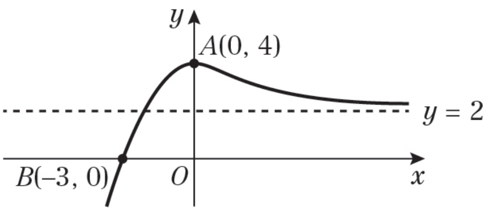
$A'(3, 2)$, $B'(0, 0)$
The asymptote is $y = 1$.

c $f(x) - 2$ is a translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, or two units down.



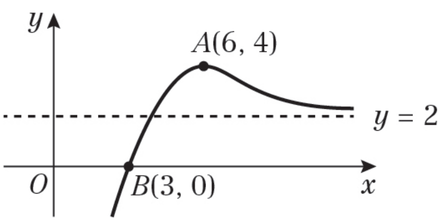
$A'(3, 2)$, $B'(0, -2)$
The asymptote is $y = 0$.

d $f(x + 3)$ is a translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, or three units to the left.



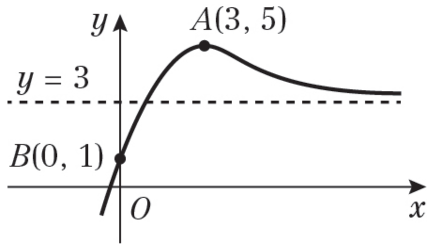
$A'(0, 4)$, $B'(-3, 0)$
The asymptote is $y = 2$.

e $f(x - 3)$ is a translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, or three units to the right.



e $A'(6, 4)$, $B'(3, 0)$
The asymptote is $y = 2$.

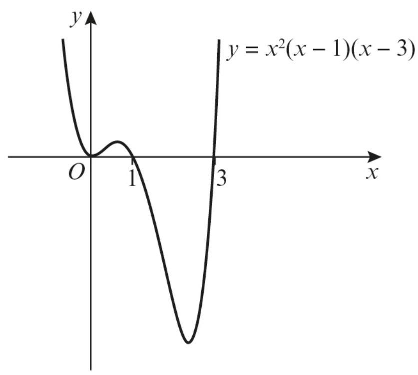
f $f(x) + 1$ is a translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or one unit up.



$A'(3, 5)$, $B'(0, 1)$
The asymptote is $y = 3$.

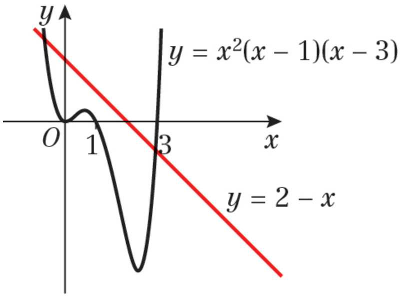
4
$$2 = 5 + 2x - x^2$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
So $x = -1$ or $x = 3$

5 a $y = x^2(x - 1)(x - 3)$
$0 = x^2(x - 1)(x - 3)$
So $x = 0$, $x = 1$ or $x = 3$
The curve touches the x-axis at $(0, 0)$ and crosses it at $(1, 0)$ and $(3, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $y = 2 - x$ is a straight line.
It crosses the x-axis at $(2, 0)$ and the y-axis at $(0, 2)$.

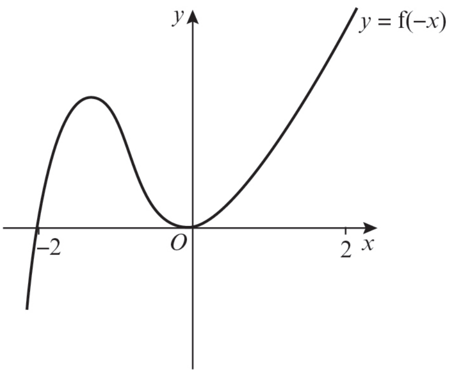
5 b



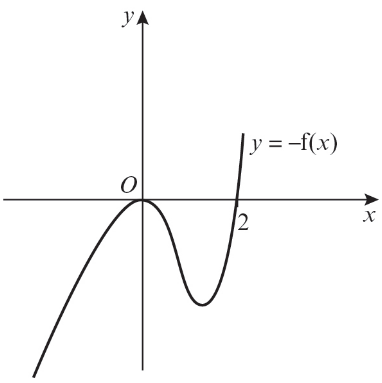
c As there are two points of intersection, $x^2(x-1)(x-3) = 2-x$ has two real solutions.

d $y = \mathrm{f}(x) + 2$ is a translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.
So $y = \mathrm{f}(x) + 2$ crosses the y-axis at $(0, 2)$.

6 a $\mathrm{f}(-x)$ is a reflection in the y-axis.

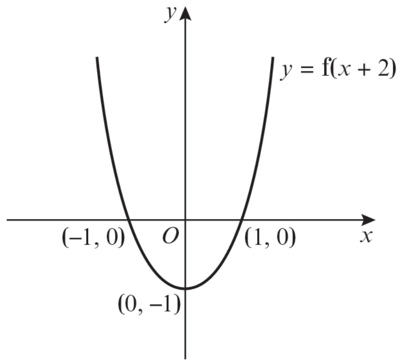


b $-\mathrm{f}(x)$ is a reflection in the x-axis.

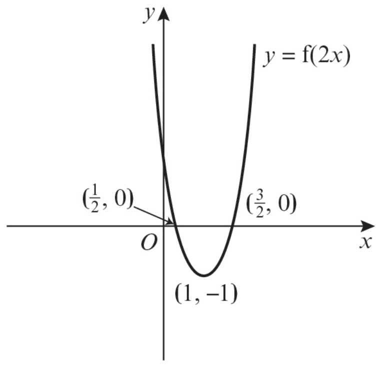


7 a Let $y = a(x-p)(x-q)$
Since $(1, 0)$ and $(3, 0)$ are on the curve then $p = 1$ and $q = 3$.
So $y = a(x-1)(x-3)$
Using $(2, -1)$:
$-1 = a(1)(-1)$
$a = 1$
So $y = (x-1)(x-3) = x^2 - 4x + 3$

b i $\mathrm{f}(x+2) = (x+1)(x-1)$, or a translation by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.



ii $\mathrm{f}(2x) = (2x-1)(2x-3)$, or a stretch with scale factor $\frac{1}{2}$ in the x-direction.



8 a $\mathrm{f}(x) = (x-1)(x-2)(x+1)$
When $x = 0$, $y = (-1) \times (-2) \times 1 = 2$
So the curve crosses the y-axis at $(0, 2)$.

b $y = a\mathrm{f}(x)$ is a stretch with scale factor a in the y-direction.
The y-coordinate has multiplied by -2, therefore $y = -2\mathrm{f}(x)$.
$a = -2$

8 c $f(x) = (x - 1)(x - 2)(x + 1)$
$0 = (x - 1)(x - 2)(x + 1)$
So $x = 1$, $x = 2$ or $x = -1$
The curve crosses the x-axis at $(1, 0)$, $(2, 0)$ and $(-1, 0)$.
$y = f(x + b)$ is a translation b units to the left.
For the point $(0, 0)$ to lie on the translated curve, either the point $(1, 0)$, $(2, 0)$ or $(-1, 0)$ has translated to the point $(1, 0)$.
For the coordinate $(1, 0)$ to be translated to $(0, 0)$, $b = 1$.
For the coordinate $(2, 0)$ to be translated to $(0, 0)$, $b = 2$.
For the coordinate $(-1, 0)$ to be translated to $(0, 0)$, $b = -1$.
$b = -1$, $b = 1$ or $b = 2$

9 a i $y = f(3x)$ is a stretch with scale factor $\frac{1}{3}$ in the x-direction. Find $\frac{1}{3}$ of the x-coordinate.
P is transformed to $\left(\frac{4}{3}, 3\right)$.

ii $\frac{1}{2}y = f(x)$
$y = 2f(x)$, which is a stretch with scale factor 2 in the y-direction.
P is transformed to $(4, 6)$.

iii $y = f(x - 5)$ is a translation by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$, or five units to the right.
P is transformed to $(9, 3)$.

iv $-y = f(x)$
$y = -f(x)$, which is a reflection of the curve in the x-axis.
$(4, -3)$

v $2(y + 2) = f(x)$
$y = \frac{1}{2}f(x) - 2$, which is a stretch with scale factor $\frac{1}{2}$ in the y-direction and then a translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, or two units down.
P is transformed to $\left(4, -\frac{1}{2}\right)$.

9 b $P(4, 3)$ is transformed to $(2, 3)$.
Either the x-coordinate has halved, which is a stretch with scale factor $\frac{1}{2}$ in the x-direction, or it has had 2 subtracted from it, which is a translation by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.

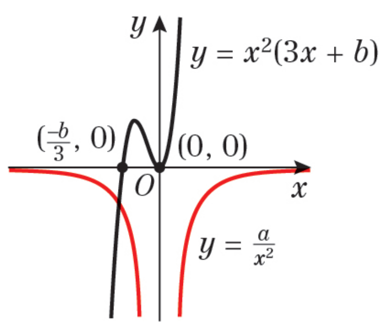
So the transformation is $y = f(2x)$ or $y = f(x + 2)$.

c i $P(4, 3)$ is translated to the point $(8, 6)$.
The x-coordinate of P has 4 added to it and the y-coordinate has 3 added to it.
$y = f(x - 4) + 3$

ii $P(4, 3)$ is stretched to the point $(8, 6)$.
The x-coordinate of P has doubled and the y-coordinate has doubled.
$y = 2f\left(\frac{1}{2}x\right)$

10 a $y = -\dfrac{a}{x^2}$ is a $y = \dfrac{k}{x^2}$ graph with $k < 0$.
x^2 is always positive and $k < 0$, so the y-values are all negative.

$y = x^2(3x + b)$
$0 = x^2(3x + b)$
So $x = 0$ or $x = -\dfrac{b}{3}$
The curve crosses the x-axis at $\left(-\dfrac{b}{3}, 0\right)$ and touches it at $(0, 0)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



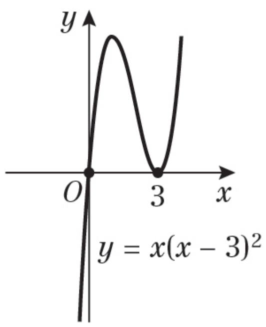
10 b From the sketch, there is only one point of intersection of the curves. This means there is only one value of x where

$$-\frac{a}{x^2} = x^2(3x+b)$$
$$-a = x^4(3x+b)$$
$$x^4(3x+b)+a=0$$

So this equation has one real solution.

11 a $x^3 - 6x^2 + 9x$
$= x(x^2 - 6x + 9)$
$= x(x-3)^2$

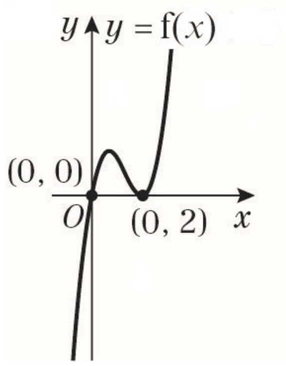
b $y = x(x-3)^2$
$0 = x(x-3)^2$
So $x = 0$ or $x = 3$
The curve crosses the x-axis at (0, 0) and touches it at (3, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



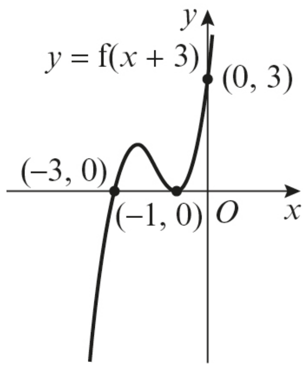
c $y = (x-k)^3 - 6(x-k)^2 + 9(x-k)$ is a translation of the curve $y = x^3 - 6x^2 + 9x$ by $\begin{pmatrix} k \\ 0 \end{pmatrix}$, or k units to the right.

For the point (−4, 0) to lie on the translated curve, either the point (0, 0) or (3, 0) has translated to the point (−4, 0).
For the coordinate (0, 0) to be translated to (−4, 0), $k = -4$.
For the coordinate (3, 0) to be translated to (−4, 0), $k = -7$.
$k = -4$ or $k = -7$

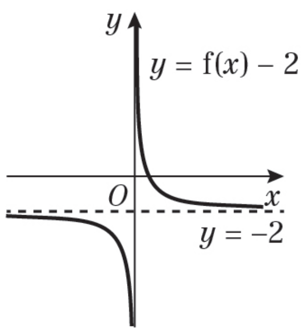
12 a $y = x(x-2)^2$
$0 = x(x-2)^2$
So $x = 0$ or $x = 2$
The curve crosses the x-axis at (0, 0) and touches it at (2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



b $y = f(x+3)$ is a translation by vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ of $y = f(x)$, or three units to the left.
So the curve crosses the x-axis at (−3, 0) and touches it at (−1, 0).
When $x = 3$, $f(x) = 3(3-2)^2$
$= 3$
So $f(x+3)$ crosses the y-axis at (0, 3).



13 a $y = f(x) - 2$ is a translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, or two units down.



13 a The horizontal asymptote is $y = -2$.
The vertical asymptote is $x = 0$.

b From the sketch, the curve crosses the x-axis.

$y = \text{f}(x) - 2$

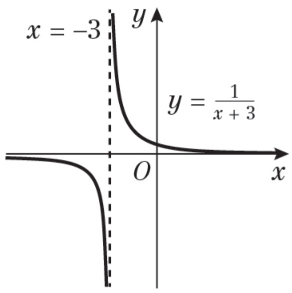
$= \frac{1}{x} - 2$

$0 = \frac{1}{x} - 2$

$x = \frac{1}{2}$

So the curve cuts the x-axis at $\left(\frac{1}{2}, 0\right)$.

c $y = \text{f}(x + 3)$ is a translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, or three units to the left.



d The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -3$.

$y = \text{f}(x + 3)$

$= \frac{1}{x+3}$

When $x = 0$, $y = \frac{1}{3}$

So the curve cuts the y-axis at $\left(0, \frac{1}{3}\right)$.

Challenge

$R(6, -4)$

$y = \text{f}(x + c) - d$ is a translation by $\begin{pmatrix} -c \\ 0 \end{pmatrix}$, or c units to the left and a translation by $\begin{pmatrix} 0 \\ -d \end{pmatrix}$, or d units down.

So R is transformed to $(6 - c, -4 - d)$.