Public Good Provision: The Lindahl-VCG Relationship*

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December 24, 2023

Abstract

We consider a public good environment in which the agents consuming the public good have quasilinear utility. We show that if the public good must be provided as a single, indivisible unit, then the VCG transfer paid by each agent is equal to the *smallest Lindahl expenditure* of this agent, while the *largest Lindahl revenue* of the firm is equal to the VCG transfer received by the firm. If the public good can be provided in multiple units, then the smallest Lindhal price of an agent is an upper bound on the agent's marginal VCG transfer for each unit on which the agent is pivotal, and the largest Lindahl revenue of the firm is a lower bound on the firm's VCG transfer. Consequently, the difference between the largest Lindahl revenue of the firm and the sum of the smallest Lindahl expenditures of each agent is equal to the VCG deficit with a single unit and a lower bound on the VCG deficit with multiple units. Our results extend to the provision of public bads such as data usage by platforms that harms consumers.

Keywords: mechanism design and Lindahl prices, public bads, complementary products, data and privacy

JEL-Classification: C72; D44; D61

*The paper has benefited from comments and feedback by seminar audiences at the Saarland Workshop in Economic Theory 2022, APIOC 2022 in Sydney, ESAM 2023 in Sydney, SAET 2023 in Paris, the University of Queensland and the University of Melbourne and from conversations with Priscilla Man. Financial support through a visiting research scholar grant from the Faculty of Business and Economics at the University of Melbourne, the Australian Research Council Grant DP200103574, and the Samuel and June Hordern Endowment is gratefully acknowledged. Mezzetti's work was funded in part by Australian Research Council grant DP190102904.

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1 Introduction

Achieving an efficient provision of public goods is complicated by the agents' incentive to free ride (Hume, 1739; Samuelson, 1954). In a setting with complete information, Lindahl prices – personalized, constant per-unit prices – induce agents to demand and the profit-maximizing firm to supply the efficient level of a public good without requiring outside funds. However, Lindahl prices are often dismissed on the grounds of being impractical because their implementation requires a firm with market power to behave as a price taker and the ability to exclude those agents who do not pay (Mas-Colell et al., 1995, p. 364) and because Lindahl prices are not constructed to provide the participants with the correct incentives to reveal the information about their marginal utilities truthfully. For example, Samuelson (1954, p.388/9) notes: "But, and this is the point sensed by Wicksell but perhaps not fully appreciated by Lindahl, now it is in the selfish interest of each person to give false signals, to pretend to have less interest in a given collective consumption activity than he really has, etc." In contrast, with quasilinear utility, the seemingly unrelated VCG mechanism, which is designed to explicitly account for the participants' private information, provides them with dominant strategies to report their preferences truthfully and induces efficient production while respecting the participants' individual rationality constraints. However, it runs a deficit.[1]

In this paper, we show that there is a tight connection between Lindahl prices and VCG transfers by studying variants of the following setup. We first assume that a profit-maximizing firm can produce a single, indivisible unit of a public good at a cost. Many agents derive value if the public good is produced, and have quasilinear utility in money. We show that the VCG transfer paid by each agent is equal to the *smallest Lindahl price* of this agent, i.e., the smallest price that the agent can pay in any Lindahl price vector. The VCG transfer that the firm receives is equal to its *largest total Lindahl price*, i.e., the largest sum of all the agent's Lindahl prices in any Lindahl price vector. Thus, the VCG deficit in

[1]The VCG mechanism derives its label from the independent contributions of Vickrey (1961), Clarke (1971) and Groves (1973). For different specifications and notions of incentive compatibility, the result that efficient public good provision is not possible without running a deficit has been shown in the literature; see, for example, Güth and Hellwig (1986), Mailath and Postlewaite (1990) and Loertscher and Mezzetti (2019). The present paper provides an independent proof for the setups it studies.

an economy with an indivisible public good is equal to the difference between the largest Lindahl total price and the sum of the smallest Lindahl prices. An important implication of our results—one that contrasts with Paul Samuelson's remarks quoted above—is that Lindahl prices *can* incentivize the agents and the firm to report their private information. However, doing so requires giving all participants their most favorable Lindahl prices, which results in a deficit if there are multiple Lindahl price vectors: the prices paid by the agents are smaller than the prices received by the firm.

When allowing the public good to be provided in multiple discrete units, we decompose each agent's VCG transfer into marginal VCG transfers paid for each unit on which the agent is pivotal, i.e., the cost imposed on others for each unit that would efficiently not be produced without that agent. An agent's VCG transfer is equal to the sum of his marginal VCG transfers over all units on which that agent is pivotal (and the VCG transfer of an agent who is not pivotal on any unit is zero). We show that each agent's smallest Lindahl price is an upper bound on the agent's marginal VCG transfer for each unit on which the agent is pivotal. Since the marginal VCG transfers are increasing in the unit on which an agent is pivotal, this implies that the smallest Lindahl price of an agent is at least as high as the marginal VCG transfer paid for the last unit on which the agent is pivotal. This in turn implies that the agent's *smallest Lindahl expenditure*, i.e., the agent's smallest Lindahl price multiplied by the number of units produced, is at least as high as the product of the units of the public good that are produced and the marginal VCG transfer for the last unit on which the agent is pivotal. We provide necessary and sufficient conditions for an agent's smallest Lindahl price to be equal to his marginal VCG transfer for a given unit. We similarly compare the firm's largest Lindahl total price with its marginal VCG transfer for each unit produced[2] and show that the difference between the largest Lindahl revenue of the firm (i.e., the firm's largest Lindahl total price multiplied by the number of units produced) and the sum of the smallest Lindahl expenditures of each agent is a lower bound on the VCG deficit.

We finally consider an environment in which a firm derives value if it produces a single unit of a public bad that imposes a cost on the agents. Mirroring our first main result, we show that the firm's smallest Lindahl total price equals the VCG transfer it pays and that

[2]The firm is pivotal on every unit produced since nothing can be produced without it.

each agent's largest Lindahl price equals the VCG transfer he receives.

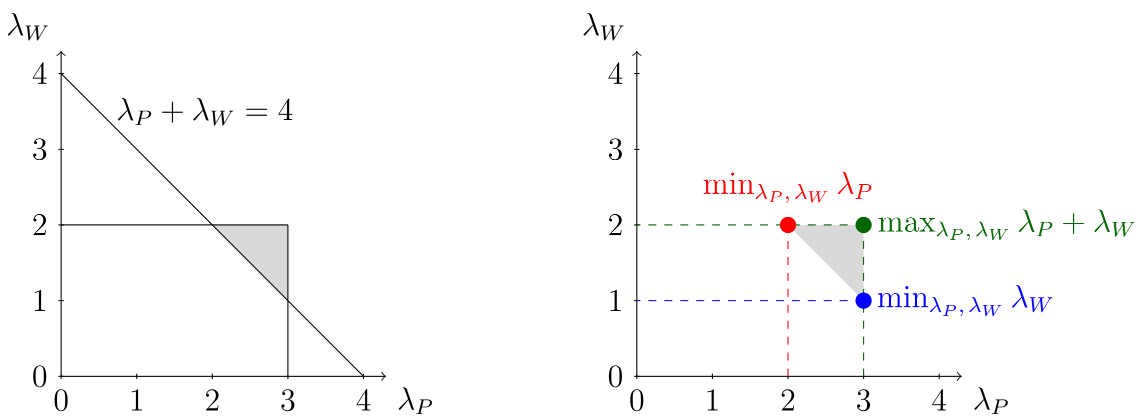
Evidently, the study of public good problems has a long tradition in economics; see, for example, Hume (1739), Wicksell (1896), Lindahl (1919), Samuelson (1954, 1955), Clarke (1971) and Green and Laffont (1977). The digital age gives renewed salience to the problem of allocating and pricing complementary goods such as intellectual property rights and consumer data (see e.g., Crémer et al., 2019). Our paper contributes to this literature by connecting the Lindahl prices that take center stage in the classic literature to the VCG transfers that arise in the incomplete information models that have come to the fore in economics after the publication of Samuelson (1954). The paper thereby parallels earlier work that, for environments with private goods and private values, has uncovered a tight connection between largest Walrasian prices and VCG transfers such as Leonard (1983), Demange (1982), Gul and Stacchetti (1999) and Loertscher and Mezzetti (2019). For exchange economies (sometimes called asset markets), this connection generalizes to an agent's largest *net* Walrasian price and its VCG transfer (Delacrétaz et al., 2022). Gul and Pesendorfer (2022) provide a cooperative foundation for Lindahl equilibrium based on weighted Nash bargaining as well as an axiomatization. Our paper complements theirs by providing a strategic foundation based on VCG.

The remainder of the paper is structured as follows. Section 2 provides an illustrative example. In Section 3, we set up the model, derive Lindahl prices and VCG transfers and spell out the specific assumptions underlying these. There we also show that the VCG mechanism always runs a deficit when production occurs. Section 4 contains the main results and Section 5 shows how the main result for a single unit extends to the case of "public bads," like those created by online platforms' usage of data that harms consumers. Section 6 concludes the paper.

2 Motivating Example

We begin with an example illustrating the link between VCG transfers and Lindahl prices. Suppose that Erik—the firm—can write a paper that Paul and William—the agents—will enjoy reading. The cost to Erik of writing the paper is 4. The paper generates a value of 3 for Paul and 2 for William. It is efficient for the paper to be written since it generates a net total value of 1 $(= 3 + 2 - 4)$.

The VCG transfer that Paul pays is the difference between the joint surplus that Erik and William would obtain if Paul were absent and the joint surplus Erik and William obtain when Paul is present. If Paul were absent, writing the paper would not be efficient as it would yield a negative social surplus $(2 - 4 = -2)$; therefore, the paper would not be written and Erik and Paul jointly obtain a value of 0. If Paul is present, the paper is written so Erik and Paul's joint surplus is -2 $(= 2 - 4)$. Therefore, Paul's VCG transfer is 2 $(= 0 - (-2))$; it is the externality that Paul's presence imposes on Erik and William. Analogously, Erik and Paul's joint surplus is 0 if William is absent (the paper is not written) and -1 $(= 3 - 4)$ if William is present. Therefore, William's VCG transfer is 1 $(= 0 - (-1))$. Finally, the VCG transfer received by Erik is the total value that his paper creates for Paul and William. If Erik is present, the paper is written so Paul and William's total value is 5 $(= 3 + 2)$. If Erik is absent, the paper is not written so Paul and William's total value of 0. Therefore, the VCG transfer received by Erik is 5 $(= 5 - 0)$. The resulting deficit for the market maker is 2 $(= 5 - 2 - 1)$.



(a) The set of Lindahl price vectors.

(b) Extremal Lindahl Prices.

Figure 1: Panel (a): The set of Lindahl price vectors (shaded). Panel (b): Extremal Lindahl prices.

Consider now the set of Lindahl prices, which is depicted in Figure 1(a). A price vector (λ_P, λ_W) consists of a price λ_P for Paul and a price λ_W for William that will be paid to Erik for writing the paper. A price vector is a Lindahl price vector if the following three conditions are met. First, λ_P must be small enough for Paul to want to purchase the paper at that price; that is, $\lambda_P \leq 3$ has to hold. Second, λ_W must be small enough for William to want to purchase the paper at that price; that is, $\lambda_W \leq 2$ has to hold. Third, the price received by Erik must be large enough to cover the cost of writing the paper; that is, we need $\lambda_P + \lambda_W \geq 4$. The set of price vectors that satisfy the three constraints—hence, the set of Lindahl price vectors—is the shaded triangle in Figure 1(a).

Next, consider Paul and William's *smallest Lindahl prices*; that is, the smallest price each of them can pay in any Lindahl price vector. The Lindahl price vector that minimizes the price paid by Paul is obtained by setting William's price as large as possible, that is, $\lambda_W = 2$, and setting Paul's price so that Erik's cost is just covered. that is, $\lambda_P = 2$ $(= 4 - 2)$. In Figure 1, that vector is the top-left corner of the shaded triangle. The smallest Lindahl price of Paul is 2, which is exactly his VCG transfer. Analogously, the Lindahl price vector that minimizes the price paid by William is obtained by setting Paul's price as large as possible, that is, $\lambda_P = 3$, and setting William's price so that Erik's cost is just covered, that is, $\lambda_P = 1$ $(= 4 - 3)$. In Figure 1, that vector is the bottom-right corner of the shaded triangle. The smallest Lindahl price of William is 1, which is exactly his VCG transfer.

Finally, let us look at Erik's *largest Lindahl total price*; that is, the largest sum of prices he can receive in any Lindahl price vector. Erik receives $\lambda_P + \lambda_W$ for writing the paper so we maximize that sum subject to the constraints that Paul and William must be willing to buy. Therefore, each price is set to the largest possible amount: $\lambda_P = 3$ and $\lambda_W = 2$. In Figure 1, that vector is the top-right corner of the shaded triangle. Erik's largest Lindahl revenue is 5, which is exactly his VCG transfer.

Our example reveals a strong relationship between Lindahl prices and VCG transfers. In the remainder of the paper, we show that this result generalizes to an arbitrary number of agents and arbitrary values and costs. When only one unit of the public good can be produced, each participant's most favorable Lindahl price or total price is equal to the participant's VCG transfer or transfer (Theorem 1). When multiple units of the public good can be produced, the equality does not hold in general; we show that each participant's marginal VCG transfer is at least as favorable as the participant's most favorable Lindahl price (Theorem 2). We also provide conditions under which the two are equal and extend our analysis to the provision of a public bad.

3 Setup

In Section 3.1, we set up the bare bones model. In Section 3.2, we then lay out the specific informational and behavioral assumptions underlying Lindahl pricing and derive the Lindahl prices. Sections 3.3 introduces the basic notions from mechanism design theory such as dominant-strategy incentive compatibility and ex post individual rationality constraints, and it derives the VCG transfers and shows that the VCG mechanism always runs a deficit when production is efficient.

3.1 Model

We consider an economy with a single firm f and a set of agents $\mathcal{N}$ with $N = |\mathcal{N}| \geq 1$. The firm can produce up to $M \geq 1$ identical and indivisible units of a public good. The marginal cost of producing the k-th unit ($k = 1, \ldots, M$) is $c^k \in [0, \overline{c}]$; therefore, the cost of producing $m \in \{1, \ldots, M\}$ units is $\sum_{k=1}^{m} c^k$. We assume *increasing marginal cost* (IMC): $0 \leq c^1 \leq c^2 \leq \ldots \leq c^M \leq \overline{c}$.

The N agents consume the public good and an aggregate good (money) and have quasi-linear utility in money. The marginal value of the k-th unit to agent $i \in \mathcal{N} = \{1, \ldots, N\}$ is denoted by $v_i^k \in [0, \overline{v}]$. Therefore, if $m \in \{1, \ldots, M\}$ units are produced, then the value to each agent i is $\sum_{k=1}^{m} v_i^k$. We assume *decreasing marginal values* (DMV), that is, $\overline{v} \geq v_i^1 \geq v_i^2 \geq \ldots \geq v_i^M \geq 0$ for every $i \in \mathcal{N}$. Further, we assume $\overline{c} \geq N\overline{v}$, which means that if the firm has sufficiently high costs, then it is efficient not to produce any unit even if all the agents have the highest possible values for every unit. We assume that the value of not participating or if no unit of the public good is produced is 0 for every agent and for the firm, and that participation is voluntary. The latter assumption means that every agent (as well as the firm) has the ability to block production by walking away.

If $m \geq 1$ units are produced, the firm is paid $\overline{t}_f$ and each agent $i \in \mathcal{N}$ pays $\underline{t}_i$, the payoffs are

$$\overline{t}_f - \sum_{k=1}^{m} c^k \qquad \text{and} \qquad \sum_{k=1}^{m} v_i^k - \underline{t}_i \quad \text{for each } i \in \mathcal{N}.$$

Producing q units of the public good is *efficient* if it maximizes the sum of the agents' marginal values minus the firm's marginal costs:

$$q \in \arg\max_{m=0,1,\ldots,M} \sum_{k=1}^{m} \left(\sum_{j \in \mathcal{N}} v_j^k - c^k \right).$$

There may be multiple efficient quantities. We denote the smallest one by q^* and refer to it as the *efficient quantity*. For $q^* \in \{1, \ldots, M\}$, by DMV and IMC, the efficient quantity q^* is characterized by the inequalities

$$\sum_{j \in \mathcal{N}} v_j^{q^*} > c^{q^*} \quad \text{and} \quad \sum_{j \in \mathcal{N}} v_j^{q^*+1} \leq c^{q^*+1},$$

where we use the convention that $c^{M+1} = \overline{c}$ and $v_j^{M+1} = 0$ for all $j \in \mathcal{N}$. If these inequalities are not satisfied, then $q^* = 0$. The maximized (*social*) *welfare* is the sum of marginal values minus marginal costs when q^* units are produced:

$$W = \sum_{k=1}^{q^*} \left(\sum_{j \in \mathcal{N}} v_j^k - c^k \right).$$

3.2 Lindahl Price Vectors

In the classic environment underlying Lindahl prices, all agents are price takers and maximize their payoffs by choosing quantities. Accordingly, the firm maximizes its profit and each agent maximizes its payoff. As with Walrasian prices, where these prices come from is not part of the model. To fix ideas, they may be thought of as being set by a fictitious and benevolent auctioneer who has complete information about the agents' values and the firms' costs.[3]

A *Lindahl price vector* is a vector of individualized unit prices $\boldsymbol{\lambda} = (\lambda_j)_{j=1,\ldots,N} \in \mathbb{R}_+^N$, one for each agent $i \in \mathcal{N}$. It must satisfy the properties that: (*i*) it is optimal for every agent $i \in \mathcal{N}$ (i.e., it maximizes i's payoff) to demand the efficient quantity q^* of the public good when the price of each unit is λ_i, and (*ii*) it is optimal for the firm (i.e., it maximizes the firm's profit) to supply the efficient quantity q^*, when each unit produced of the public good

[3]This set of assumptions is, of course, subject to a criticism that is familiar from debates related to the Walrasian model. For example, Samuelson (1955, p.354) writes that "there is something circular and unsatisfactory" about constructions like Lindahl's: "[T]hey show what the final equilibrium looks like, but by themselves they are not generally able to find the desired equilibrium."

can be sold at a total price $\sum_{j\in\mathcal{N}}\lambda_j$. Formally:

$$q^*\in\arg\max_{m=1,\dots,M}\sum_{k=1}^{m}v_i^k-m\lambda_i \text{ for every } i\in\mathcal{N}\quad\text{and}\quad q^*\in\arg\max_{m=1,\dots,M} m\sum_{j\in\mathcal{N}}\lambda_j-\sum_{k=1}^{m}c^k.$$

By DMV and IMC, those conditions simplify to

$$v_i^{q^*+1}\le\lambda_i\le v_i^{q^*} \text{ for every } i\in\mathcal{N}\quad\text{and}\quad c^{q^*}\le\sum_{j\in\mathcal{N}}\lambda_j\le c^{q^*+1}. \tag{1}$$

We denote by Λ the set of Lindahl price vectors. Observe that Lindahl price vectors satisfy complete-information analogues to incentive compatibility and individual rationality constraints insofar as they give every agent and the firm the incentive to choose the efficient quantity q^* and make participation weakly better than walking away.

Agent i's ($i\in\mathcal{N}$) *smallest Lindahl price* is the smallest price i can face in any Lindahl price vector:

$$\underline{\lambda}_i=\min_{\boldsymbol{\lambda}\in\Lambda}\lambda_i.$$

Given a Lindahl price vector $\boldsymbol{\lambda}\in\Lambda$, agent i's expenditure is what i spends for the units produced: $E_i(\lambda)=q^*\lambda_i$. Agent i's *smallest Lindahl expenditure* is the smallest expenditure i can face in any Lindahl price vector:

$$\underline{E}_i=\min_{\boldsymbol{\lambda}\in\Lambda}q^*\lambda_i=q^*\underline{\lambda}_i.$$

Agent i's smallest Lindahl price and expenditure can be expressed in terms of marginal values and costs by minimizing λ_i with respect to (1). To do so, set the price of each agent $j\in\mathcal{N}\setminus\{i\}$ to its upper bound ($\lambda_j=v_j^{q^*}$) and then set λ_i so that one of the left constraints in (1) holds with an equality. Then,

$$\underline{\lambda}_i=\max\left\{v_i^{q^*+1},c^{q^*}-\sum_{j\in\mathcal{N}\setminus\{i\}}v_j^{q^*}\right\}\quad\text{and}\quad\underline{E}_i=q^*\max\left\{v_i^{q^*+1},c^{q^*}-\sum_{j\in\mathcal{N}\setminus\{i\}}v_j^{q^*}\right\}. \tag{2}$$

Given a $\boldsymbol{\lambda}\in\Lambda$, the total price received by the firm for each unit of the public good produced is $\sum_{j\in\mathcal{N}}\lambda_j$, and the revenue when the efficient quantity is produced is $q^*\sum_{j\in\mathcal{N}}\lambda_j$. Then, letting

$$\overline{\lambda}_f=\max_{\boldsymbol{\lambda}\in\Lambda}\sum_{j\in\mathcal{N}}\lambda_j$$

be the firm's *largest Lindahl total price*, the firm's *largest Lindahl revenue* is

$$\overline{R}_f = \max_{\boldsymbol{\lambda} \in \Lambda} q^* \sum_{j \in \mathcal{N}} \lambda_j = q^* \overline{\lambda}_f.$$

The firm's largest Lindahl total price and revenue can be expressed in terms of marginal values and costs by maximizing $\sum_{j \in \mathcal{N}} \lambda_j$ with respect to (1), which gives

$$\overline{\lambda}_f = \min \left\{ \sum_{j \in \mathcal{N}} v_j^{q^*}, c^{q^*+1} \right\} \quad \text{and} \quad \overline{R}_f = q^* \min \left\{ \sum_{j \in \mathcal{N}} v_j^{q^*}, c^{q^*+1} \right\}. \tag{3}$$

3.3 VCG Transfers

The informational assumptions underlying the VCG mechanism are fundamentally different from those used in the analysis of Lindahl prices. In particular, every agent $i \in \mathcal{N}$ is now assumed to be privately informed about its type $\mathbf{v}_i = (v_i^k)_{k=1,\dots,M}$ and the firm to be privately informed about its cost $\mathbf{c} = (c^k)_{k=1,\dots,M}$. In lieu of an auctioneer who sets prices given knowledge about $\mathbf{v} = (\mathbf{v}_i)_{i \in \mathcal{N}}$ and $\mathbf{c}$, the stipulation is now that there is a mechanism designer who uses a direct mechanism $\langle q, \mathbf{t} \rangle$, where $q : [0, \overline{v}]^{NM} \times [0, \overline{c}]^M \to \{0, 1 \dots, M\}$ is the *allocation rule* and $\mathbf{t} : [0, \overline{v}]^{NM} \times [0, \overline{c}]^M \to \mathbb{R}^{N+1}$ is the *transfer rule* with, for $i \in \mathcal{N}$, t_i being the transfer from agent i to the designer and t_f being the transfer from the designer to the firm. Accordingly, given transfers $\mathbf{t}$, the designer's revenue is

$$R = \sum_{j \in \mathcal{N}} t_j - t_f.$$

The mechanism is direct insofar as it asks all agents and the firm to report their types and makes the allocation and transfers a function of these reports.[4] A mechanism is *dominant strategy incentive compatibly* (DIC) if, for all $i \in \mathcal{N}$, all $\mathbf{v}_i, \hat{\mathbf{v}}_i \in [0, \overline{v}]^M$, all $\mathbf{v}_{-i} \in [0, \overline{v}]^{(N-1)M}$ and all $\mathbf{c} \in [0, \overline{c}]^M$,

$$\sum_{k=1}^{q(\mathbf{v}_i, \mathbf{v}_{-i}, \mathbf{c})} v_i^k - t_i(\mathbf{v}_i, \mathbf{v}_{-i}, \mathbf{c}) \geq \sum_{k=1}^{q(\hat{\mathbf{v}}_i, \mathbf{v}_{-i}, \mathbf{c})} v_i^k - t_i(\hat{\mathbf{v}}_i, \mathbf{v}_{-i}, \mathbf{c})$$

holds and, for all $\mathbf{c}, \hat{\mathbf{c}} \in [0, \overline{c}]^M$ and all $\mathbf{v} \in [0, \overline{v}]^{NM}$,

$$t_f(\mathbf{c}, \mathbf{v}) - \sum_{k=1}^{q(\mathbf{v}, \mathbf{c})} c^k \geq t_f(\hat{\mathbf{c}}, \mathbf{v}) - \sum_{k=1}^{q(\mathbf{v}, \hat{\mathbf{c}})} c^k$$

[4] By the revelation principle, the focus on direct mechanisms is without loss of generality.

holds. That is, fixing the reported type profile of all others, every agent and the firm are better off reporting truthfully their types than reporting anything else. The mechanism satisfies *ex post individual rationality* (EIR) if for any possible type profile every agent is weakly better of participating than walking away (and thereby blocking any production). Because the values of the outside options are assumed to be 0, this means that for any possible type profile the expected payoff is non-negative for every $i \in \mathcal{N}$ and for the firm f. A mechanism $\langle q, \mathbf{t} \rangle$ satisfying DIC and EIR is said to be (ex post) *efficient* if it always chooses an efficient allocation, that is, if $q = q^*$.

We now derive the *VCG transfers.* The VCG mechanism is the efficient DIC-EIR mechanism that maximizes the designer's revenue R.[5] The welfare of others when agent i is present is defined as:

$$\sum_{k=1}^{q^*} \left(\sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k - c^k \right).$$

If an agent i is not present, the efficient quantity q_{-i}^* is the smallest element of

$$\arg\max_{m=0,1,\ldots,M} \sum_{k=1}^{m} \left(\sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k - c^k \right),$$

which, for $q_{-i}^* > 0$, is characterized by the inequalities

$$\sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q_{-i}^*} > c^{q_{-i}^*} \quad \text{and} \quad \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q_{-i}^*+1} \leq c^{q_{-i}^*+1}.$$

The welfare of others when i is present, which is $W - \sum_{j=1}^{q^*} v_i^j$, is weakly smaller than the welfare in an economy in which agent i is not present, which is denote by W_{-i} and given as:

$$W_{-i} = \sum_{k=1}^{q_{-i}^*} \left(\sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k - c^k \right).$$

As marginal values are nonnegative, the presence of i only increases the marginal value of every unit; therefore $q_{-i}^* \leq q_i^*$. We denote by $x_i^* = q_i^* - q_{-i}^*$ the number of units on which

[5] Holmström (1979) showed that DIC and ex post efficiency imply that the mechanism has to be a Groves' mechanism (Groves, 1973), given a smoothly connected type space, which is an assumption that is satisfied in our setting. Making the EIR constraint hold with equality for the worst-off types then means that the mechanism maximizes revenue subject to efficiency, DIC and EIR. The direct mechanism that does this is the VCG mechanism.

agent i is *pivotal*; that is, the number of units that it is no longer efficient to produce if agent $i \in \mathcal{N}$ is absent. Finally, the welfare of agents when the firm is present is:

$$\sum_{k=1}^{q^*}\left(\sum_{j\in\mathcal{N}} v_j^k\right),$$

while the welfare of agents when the firm is absent is zero.

The VCG transfer paid by agent i is

$$\underline{t}_i = \underbrace{\sum_{k=1}^{q^*_{-i}}\left(\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^k - c^k\right)}_{\text{Welfare of others when } i \text{ is not present, } W_{-i}} - \underbrace{\sum_{k=1}^{q^*}\left(\sum_{j\in\mathcal{N}\setminus\{i\}} v_j^k - c^k\right)}_{\text{Welfare of others when } i \text{ is present, } W - \sum_{k=1}^{q^*} v_i^k},$$

which simplifies to

$$\underline{t}_i = \sum_{k=q^*_{-i}+1}^{q^*}\left(c^k - \sum_{j\in\mathcal{N}\setminus\{i\}} v_j^k\right) \geq 0. \tag{4}$$

Intuitively, i's presence results in more units being produced than would be efficient for the other agents; hence i has to pay the social cost of those units to others.

The social cost of the units on which agent i is pivotal can be decomposed into the sum of the marginal social costs for each additional pivotal unit $k \in \{1, \ldots, x_i\}$. For each unit on which agent i is pivotal, define i's *marginal VCG transfer* to be

$$\underline{t}_i^k = c^{q^*_{-i}+k} - \sum_{j\in\mathcal{N}\setminus\{i\}} v_j^{q^*_{-i}+k} \quad \text{for each } k = 1, \ldots, x_i^*.$$

Note that, by the DMV and IMC assumptions, agent i's marginal VCG transfer is increasing in k. Observe also that agent i's VCG transfer is equal to the sum of i's marginal VCG transfers:

$$\underline{t}_i = \sum_{k=1}^{x_i^*} \underline{t}_i^k.$$

The *VCG transfer* that the firm receives is

$$\bar{t}_f = \underbrace{\sum_{k=1}^{q^*}\sum_{j\in\mathcal{N}} v_j^k}_{\text{Agents' welfare when } f \text{ is present}} - \underbrace{0}_{\text{Agents' welfare when } f \text{ is not present}} = \sum_{k=1}^{q^*}\sum_{j\in\mathcal{N}} v_j^k \geq 0. \tag{5}$$

Intuitively, the firm is paid for the value it creates for the agents.

Like in the case of the agents, the firm's VCG transfer can be decomposed into marginal VCG transfers. Let

$$\bar{t}_f^k = \sum_{j \in \mathcal{N}} v_j^k \quad \text{for each } k = 1, \ldots, q^*.$$

Note that the marginal VCG transfer to the firm is decreasing in k and the firm's VCG transfer is the sum of its marginal VCG transfers:

$$\bar{t}_f = \sum_{k=1}^{q^*} \bar{t}_f^k.$$

As nothing can be produced without the firm, q^* can be interpreted as the number of units on which the firm is pivotal; hence, it plays the same role as x_i^* in the formula for agents.

Note that if the public good is not produced, $q^* = 0$, then $\bar{t}_f = 0$; it is then convenient to define $\bar{t}_f^0 = 0$. To account for agents who are not pivotal on any units, it is also convenient to define $\underline{t}_i^0 = 0$ for every agent i.

Also note that the assumption that $\bar{c} \geq N\bar{v}$ ensures that the firm has veto power and can prevent the production of any unit of the public good by reporting a sufficiently high cost, thereby guaranteeing that the firm's individual rationality constraint is satisfied (its profit is at least zero) under a VCG scheme. The agents' individual rationality constraints are satisfied because each agent can guarantee at least a payoff of zero by reporting a value of zero for every unit.

Let $R^{VCG} = \sum_{i \in \mathcal{N}} \underline{t}_i - \bar{t}_f$ be the revenue of the VCG mechanism.

Lemma 1. *For any type profile, $R^{VCG} \leq 0$ holds. The inequality is strict unless $q^* = 0$.*

Proof. If $q^* = 0$, then $\bar{t}_f = 0$ and $\underline{t}_i = 0$ for all $i \in \mathcal{N}$ and thus $R^{VCG} = 0$ for $q^* = 0$.

If $q^* > 0$, to simplify notation let $V_i = \sum_{k=1}^{q^*} v_i^k$ and $V = \sum_{j \in \mathcal{N}} V_j$ and observe that $\bar{t}_f = V$. Let $\mathcal{A} = \{i : \underline{t}_i > 0\}$ be the possibly empty set of agents who are pivotal. If $\mathcal{A} = \emptyset$, then $\sum_{\in \mathcal{N}} \underline{t}_i = 0$, implying $R^{VCG} = -V < 0$, where the inequality follows because $V > 0$ (our selection of q^* as the smallest efficient quantity implies that $V > 0$ if $q^* > 0$). If $\mathcal{A} \neq \emptyset$, we have

$$\sum_{i \in \mathcal{N}} \underline{t}_i = \sum_{i \in \mathcal{A}} \underline{t}_i = \sum_{i \in \mathcal{A}} (W_{-i} - W) + \sum_{i \in \mathcal{A}} V_i$$

because $\underline{t}_i = W_{-i} - (W - V_i)$. Because $W_{-i} - W < 0$ if i is pivotal (if i were not pivotal we could have $W_{-i} - W = 0$ if $\mathbf{v}_i = \mathbf{0}$), we have

$$\sum_{i \in \mathcal{N}} \underline{t}_i < \sum_{i \in \mathcal{A}} V_i.$$

Because $\sum_{i \in \mathcal{A}} V_i \leq V = \bar{t}_f$, it follows that the transfers from the agents are strictly less than the transfer to the firm, implying $R^{VCG} < 0$ whenever $q^* > 0$, as claimed. $\square$

Lemma 1 provides a two-fold generalization of the impossibility result for public goods in Loertscher and Mezzetti (2019) to environments with decreasing marginal values and increasing marginal costs while also allowing for private information by the producer. For positive production, the latter implies that $R^{VCG} < 0$ even with a single agent (or equivalently, if $\mathbf{v}_{-i} = \mathbf{0}$) because there is an additional bilateral trade aspect à la Myerson and Satterthwaite (1983) inherent in the problem.

4 Results

In this section, we establish the connection between VCG transfers and Lindahl prices. It is useful to distinguish between the case when only a single indivisible public good can be produced and the case when multiple units can be produced.

4.1 One Unit

We first consider the case when $M = 1$; that is, the public good can only be produced as a single, indivisible unit.

Theorem 1. *Suppose that there is a single, indivisible, public good, that is, $M = 1$. Then:*

(*a*) *The smallest Lindhal expenditure of each agent i is equal to agent i's VCG transfer, $\underline{E}_i = \underline{t}_i$ for every $i \in \mathcal{N}$, and if the public good is produced, i.e. $q^* = 1$, then the smallest Lindhal price is also equal to the VCG transfer, $\underline{\lambda}_i = \underline{E}_i = \underline{t}_i$;*

(*b*) *The largest Lindahl revenue of the firm is equal to the firm's VCG transfer, $\overline{R}_f = \bar{t}_f$, and if the public good is produced, i.e. $q^* = 1$, then the largest Lindhal total price is also equal to the VCG transfer, $\overline{\lambda}_f = \overline{R}_f = \bar{t}_f$.*

Proof. As $M = 1$, $0 \leq q^*_{-i} \leq q^*_i \leq 1$. We first consider the agents and then turn to the firm.

Agents Fixing an agent $i \in \mathcal{N}$, we show that $\underline{E}_i = \underline{t}_i$, and if $q^* = 1$ then $\underline{\lambda}_i = \underline{E}_i = \underline{t}_i$. There are three possible cases to consider.

Case 1: $q^*_{-i} = q^*_i = 0$. The VCG transfer is $\underline{t}_i = \sum_{k=1}^{0} \left(c^k - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k \right) = 0$ and the smallest Lindahl expenditure is $\underline{E}_i = 0 \cdot \max \left\{ v_i^1, c^0 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^0 \right\} = 0$; hence $\underline{E}_i = \underline{t}_i$.

Case 2: $q^*_{-i} = q^* = 1$. The VCG transfer is $\underline{t}_i = \sum_{k=2}^{1} \left(c^k - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k \right) = 0$ and the smallest Lindahl expenditure is $\underline{E}_i = 1 \cdot \underline{\lambda}_i = E_i = 1 \cdot \max \left\{ v_i^2, c^1 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1 \right\} = \max \left\{ 0, c^1 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1 \right\}$. As $q^*_{-i} = 1$, we have that $c^1 \leq \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1$ and hence $\underline{\lambda}_i = \underline{E}_i = 0 = \underline{t}_i$.

Case 3: $q^*_{-i} = 0$ and $q^* = 1$. The VCG transfer is $\underline{t}_i = \sum_{k=1}^{1} \left(c^k - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k \right) = c^1 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1$. The smallest Lindahl expenditure is $\underline{E}_i = 1 \cdot \underline{\lambda}_i = 1 \cdot \max \left\{ v_i^2, c^1 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1 \right\} = \max \left\{ 0, c^1 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1 \right\}$. As $q^*_{-i} = 0$, we have that $c^1 \geq \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1$ and hence $\underline{\lambda}_i = \underline{E}_i = c^1 - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^1 = \underline{t}_i$.

Firm We show that $\overline{R}_f = \overline{t}_f$ and if $q^* = 1$ then $\overline{\lambda}_f = \overline{R}_f = \overline{t}_f$. There are two cases that we need to consider.

Case 1: $q^* = 0$. The VCG transfer is $\overline{t}_f = \sum_{k=1}^{0} \sum_{j \in \mathcal{N}} v_j^k = 0$ and the largest Lindahl revenue is $\overline{R}_f = 0 \cdot \min \left\{ \sum_{j \in \mathcal{N}} v_j^0, c^1 \right\} = 0 = \overline{t}_f$.

Case 2: $q^* = 1$. The VCG transfer is $\overline{t}_f = \sum_{k=1}^{1} \sum_{j \in \mathcal{N}} v_j^k = \sum_{j \in \mathcal{N}} v_j^1$. The largest Lindahl revenue is $\overline{R}_f = 1 \cdot \overline{\lambda}_f = 1 \cdot \min \left\{ \sum_{j \in \mathcal{N}} v_j^1, c^2 \right\}$. As $c^2 = \infty$, it follows that $\overline{\lambda}_f = \overline{R}_f = \sum_{j \in \mathcal{N}} v_j^1 = \overline{t}_f$. □

Theorem 1 shows that there is a close connection between Lindahl prices and VCG transfers: Each participant's VCG transfer is equal to his expenditure or its revenue in whatever Lindahl price vector is most favorable. For agents, this means paying the smallest possible price to the firm for the public good. For the firm, this means receiving the largest possible prices from each of the agents. Thus, a key insight from Theorem 1 is that with a single, indivisible public good, Lindahl prices *can* provide participants with an incentive to reveal their private information. However, this requires selecting each participant's most favorable Lindahl price vector. This result also sheds new light on the deficit that the market maker incurs in the VCG mechanism (see Lemma 1). The deficit arises because the prices that participants face come from different Lindahl price vectors; the prices paid by the agents must be as low as possible, and the prices collected by the firm must be as high as possible.

4.2 Multiple Units

We now allow multiple units of the public good to be produced; that is, we drop the assumption that $M = 1$. Recall that, for each buyer $i \in \mathcal{N}$, $x_i^* = q^* - q_{-i}^*$ is the number of units on which i is pivotal; that is, the number of units that would not be efficient to produce if agent i were not present. The firm, on the other hand, is pivotal on the efficient number of units q^*, because, evidently, without the firm no units of the public good could be produced.

We first provide an example to show that the equality uncovered in Theorem 1 breaks down and then proceed to show that an inequality remains true. Suppose that the firm can produce up to two units of a public good, facing a marginal cost of 3 for each unit. There are two agents: Agent 1's marginal values for the first and the second units are 3 and 2, respectively. Agent 2's marginal value for each unit is 2. The marginal costs and values are summarized below:

$$\begin{array}{cc} & \begin{array}{cc} \text{1-st unit} & \text{2-nd unit} \end{array} \\ \begin{array}{c} \text{firm} \\ \text{agent 1} \\ \text{agent 2} \end{array} & \left(\begin{array}{cc} 3 & 3 \\ 3 & 2 \\ 2 & 2 \end{array} \right) \end{array}.$$

The welfare created is 0 if nothing is produced, 2 $(= 5 - 3)$ if one unit is produced, and 3 $(= 3 + 2 + 2 + 2 - 3 - 3)$ if two units are produced. Therefore, the efficient quantity is 2. The total value to agent 1 is 5, the total value to agent 2 is 4, and the total cost to the firm is 6.

If either agent is absent, the efficient quantity falls to 0. Therefore, both agents are pivotal on two units of the public good. The marginal VCG transfer of agent 1 is 1 $(c^1 - v_2^1 = 3 - 2)$ for the first unit and 1 $(c^2 - v_2^2 = 3 - 2)$ for the second unit; the marginal transfer of agent 2 is 0 $(c^1 - v_1^1 = 3 - 3)$ for the first unit and 1 $(c^2 - v_1^2 = 3 - 2)$ for the second unit; the marginal VCG transfer received by the firm is 5 $(= v_1^1 + v_2^1 = 3 + 2)$ for the first unit and 4 $(v_1^2 + v_2^2 = 2 + 2)$ for the second unit.

For a personalized price vector (λ_1, λ_2) to be a Lindahl price vector, it must be that (i) agent 1 wants to purchase two units (i.e., $\lambda_1 \leq 2$), (ii) agent 2 wants to purchase two units (i.e., $\lambda_2 \leq 2$), and (iii) the firm wants to produce two units (i.e., $\lambda_1 + \lambda_2 \geq 3$). The Lindahl price paid by agent 1 is minimized by setting λ_2 as high as possible, i.e., $\lambda_2 = 2$, and setting λ_1 to cover the firm's cost, $\lambda_1 = 1$ $(= 3 - 2)$. Therefore, agent 1's smallest Lindahl price is 1 and his smallest Lindahl expenditure is 2. Analogous calculations yield the same smallest Lindahl price and expenditure of 1 and 2 for agent 2. The total price received by the firm for each unit of the public good is $\lambda_1 + \lambda_2$, which is maximized by setting both λ_1 and λ_2 to their largest possible values, which is 2 for both prices. Therefore, the largest Lindahl total price and revenue for the firm are 4 and 8, respectively. The most favorable Lindahl prices (smallest Lindahl price for each agent and largest Lindahl total price for the firm) and the marginal VCG transfers are summarized below:

$$\begin{array}{l} \\ \text{firm} \\ \text{agent 1} \\ \text{agent 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{best Lindahl price} & \begin{array}{c}\text{marginal VCG transfer/}\\ \text{transfer on first unit}\end{array} & \begin{array}{c}\text{marginal VCG transfer/}\\ \text{transfer on second unit}\end{array} \end{array} \\ \left(\begin{array}{ccc} 4 & 5 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right). \end{array}$$

We can see that the equality of the smallest Lindahl price and the marginal VCG transfer for an agent found in Theorem 1 holds for all units for agent 1, but only holds for the second unit for agent 2. The equality of the firm's largest Lindahl total price and the firm marginal VCG transfer holds for the second unit, but not the first. Moreover, we can see that whenever a participant's marginal VCG transfer differs from the most favorable Lindahl price, the marginal VCG transfer is better for the participant: it is smaller for agent 2 and larger for the firm. The intuitive reason is that Lindahl prices are linear; they apply to all units of the public good and thus only depend on the last unit—prices have to be low enough for each agent to buy the last unit but high enough for the firm to produce the last unit. Marginal VCG transfers, in contrast, are non-linear; they depend on values and costs associated with units other than the last one and, as a result, increase in the number of units for agents and decrease for the firm.

We now derive a series of results that generalize the insights from the example.

Theorem 2. *Suppose that there are $M \geq 1$ units of the public good. Then:*

$(a.1)$ $\underline{\lambda}_i \geq \underline{t}_i^k$, *for each* $k = 0, 1, \ldots, x_i^*$;

$(a.2)$ *If agent $i \in \mathcal{N}$ is not pivotal on any unit ($x_i^* = 0$) of the public good, then agent i's smallest Lindahl expenditure is at least as high as q^* times agent i's VCG transfer: $\underline{E}_i \geq q^* \cdot \underline{t}_i = 0$, for every $i \in \mathcal{N}$;*

$(a.3)$ *If agent $i \in \mathcal{N}$ is pivotal on at least one unit of the public good ($x_i^* \geq 1$), then agent i's smallest Lindahl expenditure is at least as high as q^* times agent i's marginal VCG transfer on the last unit on which i is pivotal, x_i^*, which in turn is at least as high as $\frac{q^*}{x_i^*}$ times agent i's VCG transfer: $\underline{E}_i \geq q^* \cdot \underline{t}_i^{x_i^*} \geq \frac{q^*}{x_i^*} \cdot \underline{t}_i$, for every $i \in \mathcal{N}$;*

$(b.1)$ *The largest Lindahl revenue of the firm is at most as large as the firm's VCG transfer: $\overline{R}_f \leq \overline{t}_f$.*

$(b.2)$ *If at least one unit of the public good is produced, $q^* \geq 1$, then the largest Lindahl total price is at most as large as each of the firm's marginal VCG transfers: $\overline{\lambda}_f \leq \overline{t}_f^k$ for each $k = 1, \ldots, q^*$.*

Proof. We first consider the agents and then turn to the firm.

Agents Fixing an agent $i \in \mathcal{N}$, we show that parts $(a.1)$–$(a.3)$ hold. We need to consider two cases.

Case 1: $x_i^* = 0$. The case assumption implies that $q_{-i}^* = q^*$; hence, the only marginal VCG transfer of agent i is $\underline{t}_i^0 = 0$ and the VCG transfer is $\underline{t}_i = \sum_{k=q_{-i}^*+1}^{q^*} \left(c^k - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k\right) = 0$. It follows that $\underline{\lambda}_i \geq \underline{t}_i^0 = \underline{t}_i$ and $\underline{E}_i = q^* \cdot \underline{\lambda}_i \geq = q^* \underline{t}_i = 0$.

Case 2: $x_i^* > 0$. The smallest Lindahl price is

$$\underline{\lambda}_i = \max\left\{v_i^{q^*+1}, c^{q^*} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*}\right\} \geq \left(c^{q^*} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*}\right) \tag{6}$$

By DMV and IMC, we have that for all $k = 1, \ldots, x_i^*$

$$\left(c^{q^*} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*}\right) \geq \left(c^{q-i+k} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q-i+k}\right) = \underline{t}_i^k. \tag{7}$$

Combining (6) and (7), it then follows that $\underline{\lambda}_i \geq \underline{t}_i^k$ for all $k = 0, 1, ..., x_i^*$ (recall that, be definition, $t_i^0 = 0$). To show that $\underline{E}_i \geq q^* \cdot \underline{t}_i^{x_i^*} \geq \frac{q^*}{x_i^*} \underline{t}_i$, note that

$$\underline{E}_i = \cdot q^* \cdot \underline{\lambda}_i \geq \cdot q^* \cdot \left(c^{q^*} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*} \right) = q^* \cdot \underline{t}_i^{x_i^*} \geq \frac{q^*}{x_i^*} \cdot \sum_{k=q_{-i}^*+1}^{q^*} \left(c^k - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^k \right) = \frac{q^*}{x_i^*} \cdot \underline{t}_i.$$

Firm The largest Lindahl total price of the firm is

$$\overline{\lambda}_f = \min \left\{ \sum_{j \in \mathcal{N}} v_j^{q^*}, c^{q^*+1} \right\} \leq \sum_{j \in \mathcal{N}} v_j^{q^*}. \tag{8}$$

By DMW, we have that, for all $k = 1, \ldots, q^*$,

$$\sum_{j \in \mathcal{N}} v_j^{q^*} = \overline{t}_f^{q^*} \leq \sum_{j \in \mathcal{N}} v_j^k = \overline{t}_f^k. \tag{9}$$

It follows from (8) that if $q^* \geq 1$, then $\overline{\lambda}_f \leq \overline{t}_f^{q^*} \leq \overline{t}_f^k$ for all $k = 1, \ldots, q^*$.

If $q^* \geq 1$, then combining (8) and (9) yields $\overline{\lambda}_f \leq \overline{t}_f^k$.

By DMW, we also have that

$$q^* \cdot \sum_{j \in \mathcal{N}} v_j^{q^*} \leq \sum_{k=1}^{q^*} \sum_{j \in \mathcal{N}} v_j^k = \overline{t}_f. \tag{10}$$

Combining (8) and (10) yields $\overline{R}_f = q^* \cdot \overline{\lambda}_f \leq \overline{t}_f$. □

We now discuss Theorem 2, starting with agents. If agent i is not pivotal on any unit (i.e., $x_i^* = 0$), then agent i's VCG transfer is zero. It is then immediate that agent i's smallest Lindahl price and smallest Lindahl expenditure are at least as high as the VCG transfer.

If agent i is pivotal on at least one unit (i.e., $x_i^* \geq 1$), then the marginal VCG transfer of agent i on the k-th unit produced in addition to what would be efficient for a society in which i were not present is equal to the externality that producing that additional unit imposes on others. The smallest Lindahl price of agent i must be at least as high as each of the marginal VCG transfers; that is, at least as high as the highest marginal VCG transfer, which is $\underline{t}_i^{x_i^*}$, the marginal VCG transfer on the last unit on which agent i is pivotal. Intuitively, as agent i must pay the same Lindahl price on each unit of the public good, every possible such a price must be at least as high as the externality imposed on others by producing any of the additional units on which agent i is pivotal, $\underline{\lambda}_i \geq t_i^k$ for each $k = 1, \ldots, x_i^*$. Thus, in particular, the smallest Lindahl expenditure of agent i, $\underline{E}_i$, must be at least as high as q^* times $\underline{t}_i^{x_i^*}$, the highest marginal VCG transfer of agent i, which in turn, by DMV and IMC, is at least as high as agent i's VCG transfer divided by the number of units x_i^* on which i is pivotal.

The firm is pivotal on all units that are produced. If at least one unit is produced, then the marginal VCG transfer received by the firm for the k-th unit produced, $\bar{t}_f^k$, is equal to the positive externality on agents (the sum of their marginal values) for that unit. The largest Lindahl total price paid to the firm, $\overline{\lambda}_f$, must be at most as high as such a positive externality on agents for any unit k. As the firm is paid the total Lindahl price on each unit produced, it follows that the firm's largest Lindahl revenue is at most as large as the firm's VCG transfer, which equals the sum over all units produced of the positive externalities on agents due to the presence of the firm.

Our public good model could be expressed as a private good model with N goods, one for each agent, each of which can be produced in up to M units, (Mas-Colell et al., 1995, p.363/4). With this transformation, results in Segal and Whinston (2016) and Delacrétaz et al. (2022) imply that each agent's smallest Lindahl expenditure is an upper bound for the VCG transfer that the agent pays and that the firm's largest Lindahl revenue is a lower bound for the VCG transfer the firm receives. Theorem 2 goes beyond that inequality and is not implied by any existing result in the private good environment.

We next identify when the agents' smallest Lindahl prices, expenditures, and VCG transfers satisfy the weak inequalities from Theorem 2 as equalities. We first consider the case when agent i is not pivotal on any unit of the public good, and then the case when i is pivotal on at least one unit.

Proposition 1. *Suppose agent $i \in \mathcal{N}$ is not pivotal on any unit of the public good ($x_i^* = 0$). Then:*

(a) The smallest Lindhal price of agent i is equal to agent i's VCG transfer, $\underline{\lambda}_i = \underline{t}_i$, if and only if $v_i^{q^*+1} = 0$.

(*b*) *The smallest Lindahl expenditure of agent* i *is equal to agent* i*'s VCG transfer,* $\underline{E}_i = \underline{t}_i$*, if and only if either* $q^* = 0$*, or* $v_i^{q^*+1} = 0$.

Proof. Observe that if agent $i \in \mathcal{N}$ is not pivotal on any unit, $x_i^* = 0$, then agent i's VCG transfer is zero, $\underline{t}_i = 0$. It follows from $q^* = q_{-i}^*$ and (2) that $\underline{\lambda}_i = 0$ if and only if $v_i^{q^*+1} = 0$ and $\underline{E}_i = q^* \cdot \underline{\lambda}_i = 0$ if and only if either $q^* = 0$, or $v_i^{q^*+1} = 0$. □

Proposition 1 is straightforward. When agent i is not pivotal on any units, the agent's VCG transfer is zero. For the smallest Lindahl price of agent i to also be zero, it must be the case that one additional unit of the public good above q^* has zero marginal value for agent i. The smallest Lindahl expenditure of agent i is of course zero if the smallest Lindahl price is zero. It is also zero if no amount of the public good is produced, irrespective of what i's Lindahl price is.

Proposition 2. *Suppose agent* $i \in \mathcal{N}$ *is pivotal on at least one unit of the public good* $(x_i^* \geq 1)$*. Then, for any* $k \in \{1, \ldots, x_i^*\}$*, the smallest Lindhal price of agent* $i \in \mathcal{N}$ *is equal to agent* i*'s marginal VCG transfer for the* k*-th unit,* $\underline{\lambda}_i = \underline{t}_i^k$ *if and only if*

(*a*) $v_i^{q^*+1} + \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*} \leq c^{q^*}$,

(*b*) $v_j^{q_{-i}+k} = v_j^{q^*}$ *for each* $j \in \mathcal{N} \setminus \{i\}$*, and*

(*c*) $c^{q_{-i}^*+k} = c^{q^*}$.

Furthermore, agent i*'s smallest Lindahl expenditure is equal to* $\frac{q^*}{x_i^*}$ *times agent* i*'s VCG transfer,* $\underline{E}_i = \frac{q^*}{x_i^*} \cdot \underline{t}_i$*, if and only if condition* (*a*) *holds and conditions* (*b*) *and* (*c*) *hold for all* $k \in \{1. \ldots, x_i^*\}$*. Finally,* $\underline{E}_i = \underline{t}_i$ *if and only, in addition to* (*a*) *holding and* (*b*) *and* (*c*) *holding for all* $k \in \{1. \ldots, x_i^*\}$*,* $x_i^* = q^*$ *also holds.*

Proof. We show the "if" and "only if" parts of the statement separately.

(If) Suppose that (*a*) holds and (*b*) and (*c*) hold for a $k \in \{1, \ldots, x_i^*\}$. First note that (*a*) ensures that agent i is pivotal on at least one unit (i.e., $x_i^* > 0$), since lowering i's marginal value for the q^*-th unit from $v_i^{q^*}$ to $v_i^{q^*+1}$ makes it socially efficient not to produce the q^*-th unit. In addition, (*a*) implies that (6) holds as an equality: $\underline{\lambda}_i = \left(c^{q^*} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*}\right)$. Conditions (*b*) and (*c*) imply that (7) holds as an equality. Hence $\underline{\lambda}_i = \underline{t}_i^k$.

If (a) holds and (b) and (c) hold for all $k \in \{1, \ldots, x_i^*\}$, then $\underline{\lambda}_i = \underline{t}_i^k$ for all $k \in \{1, \ldots, x_i^*\}$ and, as a consequence, $\underline{\lambda}_i = \frac{1}{x_i^*} \cdot \sum_{k=1}^{x_i^*} \underline{t}_i^k = \frac{1}{x_i^*} \cdot \underline{t}_i$. It follows that $\underline{E}_i = q^* \cdot \underline{\lambda}_i = \frac{q^*}{x_i^*} \cdot \underline{t}_i$.

Finally, it is immediate that if $x_i^* = q^*$ also holds, then $\underline{E}_i = \underline{t}_i$.

(Only If) Assume that $x_i^* > 0$. We show that if one of the conditions (a), (b), (c) is violated for a $k \in \{1, \ldots, x_i^*\}$, then $\underline{\lambda}_i > \underline{t}_i^k$.

If (a) is violated, then (6) holds as a strict inequality: $\underline{\lambda}_i > \left(c^{q^*} - \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*}\right) = \underline{t}_i^{x_i^*} \geq \underline{t}_i^k$ for all $k \in \{1, \ldots, x_i^*\}$, where the last inequality follows from $\underline{t}_i^k$ being increasing in k.

Note that for either (b) or (c) to be violated, then it must be the case that $x_i^* > 1$ (and $k > 1$), because (b) or (c) hold by definition for $x_i^* = 1 = k$. If either (b) or (c) does not hold, then (7) holds as a strict inequality. Hence $\underline{\lambda}_i > \underline{t}_i^k$.

If either condition (a) is violated, or (b), (c) do not hold for all $k \in \{1, ..., x_i^*\}$, then $\underline{\lambda}_i > \underline{t}_i^k$ for some k and hence $\underline{\lambda}_i > \frac{1}{x_i^*} \cdot \sum_{k=1}^{x_i^*} \underline{t}_i^k = \frac{1}{x_i^*} \cdot \underline{t}_i$. Thus, we have $\underline{E}_i = q^* \cdot \underline{\lambda}_i > \frac{q^*}{x_i^*} \cdot \underline{t}_i$.

Finally, since $\underline{E}_i \geq \frac{q^*}{x_i^*} \cdot \underline{t}_i$, it is immediate that if $1 \leq x_i^* < q^*$, then $\underline{E}_i > \underline{t}_i$. $\square$

Proposition 2 shows that if an agent is pivotal on at least one unit, then the smallest Lindhal price of that agent is equal to the agent's marginal VCG transfer for the k-th pivotal unit when three conditions are satisfied. First, i's marginal value for the first additional unit of the public good above the efficient level (i.e., the $q^* + 1$-st unit) must be sufficiently low so that, if that were the marginal value of agent i for the last efficiently produced unit (i.e., the q^*-th unit), then it would not be any longer efficient to produce that last unit. Second, the marginal values of all the other agents for the last $x_i^* - k + 1$ units of the public good that are produced (i.e., from the $q_{-i}^* + k$-th unit to the q^*-th unit) must be constant. Third, the firm's marginal costs from the $q_{-i}^* + k$-th to the q^*-th unit of the public good must be constant.

Proposition 2 also shows that if an agent is pivotal on at least one unit, then the smallest Lindhal expenditure of the agent is equal to $\frac{q*}{x_i^*}$ times the agent's VCG transfer when the three mentioned conditions hold for all efficiently produced units of the public good from the $q_{-i}^* + 1$-st to the q^*-th. As an agent cannot be pivotal on more units than are efficiently produced, Proposition 2 and Proposition 1 together imply that an agent's smallest Lindahl expenditure may only be equal to the agent's VCG transfer if the agent is pivotal on either all or none of the units that are efficiently produced. This is spelled out in the following corollary.

Corollary 1. *For every $i = 1, \ldots, N$, $\underline{E}_i = \underline{t}_i$ if and only if either*

(*a*) $x_i^* = 0$ *and* $\min\{q^*, v_i^{q^*+1}\} = 0$, *or*

(*b*) $x_i^* = q^* \geq 1$, *and conditions* (*a*), (*b*) *and* (*c*) *of Proposition 2 hold for* $k = 1$.

Proof. Part (*a*) is equivalent to part (*b*) of Proposition 1. Part (*b*) is equivalent to the last statement in Proposition 2. □

It is easy to see that the conditions for Corollary 1 are satisfied when $M = 1$ and hence part (*a*) of Theorem 1 is implied by Corollary 1. Either i is not pivotal and $x_i^* = 0$ or i is pivotal and $x_i^* = q^* = 1$. In the former case, either $q^* = 0$ or $q^* = 1$, in which case $v_i^{q^*+1} = v_i^2 = 0$. In the latter case, $v_i^{q^*+1} + \sum_{j \neq i} v_j^{q^*} \leq c^{q^*}$ simplifies to $\sum_{j \neq i} v_j^1 \leq c^1$, which is satisfied since i is pivotal. Moreover, $c^{q^*_{-i}+1} = c^{q^*}$ and $v_j^{q_{-i}+1} = v_j^{q^*}$ for all $j = 1, \ldots, N$ are trivially satisfied since $q^*_{-i} + 1 = q^* = 1$.

Corollary 1 also provides the basis to generalize part (*a*) of Theorem 1 to an environment with multiple units, constant marginal values and costs, as demonstrated by the next corollary.

Corollary 2. *Suppose that $c^1 = c^M$ and $v_i^1 = v_i^M$ for all $i \in \mathcal{N}$. Then, $\underline{E}_i = q^* \cdot \underline{\lambda}_i = \underline{t}_i$ for all $i \in \mathcal{N}$.*

Proof. By DMW and IMC, $c^1 = c^M$ and $v_i^1 = v_i^M$ for all $i \in \mathcal{N}$ imply that the marginal values of all agents and the marginal costs are constant. Let $v_i = v_i^1 = \ldots = v_i^M$ for each $i \in \mathcal{N}$ and let $c = c^1 = \ldots = c^M$. By definition, we have that

$$q^* = \begin{cases} 0 & \text{if } \sum_{j \in \mathcal{N}} v_j \leq c \\ M & \text{if } \sum_{j \in \mathcal{N}} v_j > c \end{cases} \quad \text{and} \quad q^*_{-i} = \begin{cases} 0 & \text{if } \sum_{j \in \mathcal{N} \setminus \{i\}} v_j \leq c \\ M & \text{if } \sum_{j \in \mathcal{N} \setminus \{i\} \neq i} v_j > c \end{cases},$$

which implies that

$$x_i^* = \begin{cases} 0 & \text{if} \quad \sum_{j \in \mathcal{N}} v_j \leq c \text{ or } \sum_{j \in \mathcal{N} \setminus \{i\}} v_j > c \\ M & \text{if} \quad \sum_{j \in \mathcal{N} \setminus \{i\}} v_j \leq c < \sum_{j \in \mathcal{N}} v_j \end{cases}.$$

We consider the cases where $x_i^* = 0$ and $x_i^* = M$ separately.

Case 1: $x_i^* = 0$. If $q^* = 0$, then $\underline{E}_i = q^* \cdot \underline{\lambda}_i = \underline{t}_i = 0$. If $q^* = M$, then $v_i^{q^*+1} = v_i^{M+1} = 0$, $\sum_{j \in \mathcal{N} \setminus \{i\}} v_j \leq c$ and Corollary 1 imply that $\underline{E}_i = \underline{\lambda}_i = \underline{t}_i = 0$.

Case 2: $x_i^* = M$. The case assumption directly implies that $q_{-i}^* = 0$ and $q^* = M = x_i^*$. The condition that $v_i^{q^*+1} + \sum_{j \in \mathcal{N} \setminus \{i\}} v_j^{q^*} \leq c^{q^*}$ simplifies to $\sum_{j \in \mathcal{N} \setminus \{i\}} v_j \leq c$, which is satisfied since $q_{-i} = 0$. The conditions that $c^{q^*_{-i}+1} = c^{q^*}$ and $v_j^{q^*_{-i}+1} = v_j^{q^*}$ for all $j \in \mathcal{N}$ simplify to $c^1 = c^M$ and $v_j^1 = v_j^M$ for all $j \in \mathcal{N}$. These conditions are satisfied by assumption. Then, Corollary 1 implies that $\underline{E}_i = q^* \cdot \underline{\lambda}_i = \underline{t}_i$. □

We now consider the firm and identify the conditions under which the firm's largest Lindahl total price, revenue, and VCG transfer satisfy the weak inequalities from Theorem 2 as equalities.

Proposition 3. *If the firm is not pivotal on any units, i.e., $q^* = 0$, then the largest Lindhal revenue of the firm is equal to the firm's VCG transfer, $\overline{R}_f = \overline{t}_f = 0$.*

If the firm is pivotal on at least one unit of the public good, i.e., $q^* \geq 1$, then, for any $k \in \{1, \dots, q^*\}$, the largest Lindhal total price of the firm is equal to the firm's marginal VCG transfer for the k-th unit, $\overline{\lambda}_f = \overline{t}_f^k$, if and only if

(*a*) $\quad \sum_{j \in \mathcal{N}} v_j^{q^*} \leq c^{q^*+1}$,

(*b*) $\quad v_j^k = v_j^{q^*}$ *for all* $j \in \mathcal{N}$

Furthermore, if the firm is pivotal on at least one unit of the public good, then the firm's largest Lindahl revenue is equal to the firm's VCG transfer, $\overline{R}_f = \overline{t}_f$, if and only if condition (a) holds and condition (b) holds for $k = 1$. When this is the case, it is also the case that the firm's largest Lindahl total price is equal to the firm's marginal VCG of the first unit of the public good, $\overline{\lambda}_f = \overline{t}_f^1$.

Proof. If the firm is not pivotal on any unit, $q^* = 0$, then it is immediate that $\overline{R}_f = \overline{t}_f = 0$, since $\overline{R}_f = q^* \cdot \overline{\lambda}_f$ and the VCG transfer is zero.

We show the "if" and "only if" parts of the statement for the case $q^* \geq 1$ separately.

(If) Condition (a) and (8) imply that $\overline{\lambda}_f = \sum_{j \in \mathcal{N}} v_j^{q^*}$. Condition ($b$) implies that $\sum_{j \in \mathcal{N}} v_j^k = \sum_{j \in \mathcal{N}} v_j^{q^*}$. Since $\overline{t}_f^k = \sum_{j \in \mathcal{N}} v_j^k$, it follows that $\overline{\lambda}_f = \overline{t}_f^k$. Furthermore, if condition (b) holds for $k = 1$, then $\overline{\lambda}_f = \overline{t}_f^k$ for all $k \in \{1, \dots, q^*\}$. Hence $\overline{R}_f = q^* \cdot \overline{\lambda}_f = \overline{t}_f$.

(Only if) If condition (a) is violated, then (8) implies that $\overline{\lambda}_f = c^{q^*+1} < \sum_{j\in\mathcal{N}} v_j^{q^*} \leq \sum_{j\in\mathcal{N}} v_j^k = t_f^k$, where the last inequality follows from DMW.

If condition (b) is violated, then there exist $j \in \mathcal{N}$ such that $v_j^k \neq v_j^{q^*}$. Since, by DMW, $v_j^k \geq v_j^{q^*}$ for all $j \in \mathcal{N}$, it follows that $\bar{t}_f^k = \sum_{j\in\mathcal{N}} v_j^k > \sum_{j\in\mathcal{N}} v_j^{q^*} \geq \overline{\lambda}_f$, where the last inequality follows from (8).

Furthermore, if condition (b) is violated for $k = 1$, then $\bar{t}_f^1 > \overline{\lambda}_f$, and since $\bar{t}_f^1 > \overline{\lambda}_f$ it must be $\bar{t}_f = \sum_{k=1,\ldots,q^*} \bar{t}_f^k > q^* \cdot \overline{\lambda}_f = \overline{R}_f$. □

Proposition 3 shows that if it is efficient to produce a positive number of units of the public good, then the largest Lindhal total price of the firm is equal to the firm's marginal VCG transfer for the k-th unit when two conditions are satisfied. First, the marginal cost of producing one additional unit of the public above the efficient level (i.e., the $q^* + 1$-st unit) must be sufficiently high so that, if that were the marginal cost of the last efficiently produced unit (i.e., the q^*-th unit), then it would not be any longer efficient to produce that last unit. Second, the marginal values of all agents for the k-th unit of the public good must be equal to the marginal values of the last efficiently produced unit (i.e., q^*-th unit); this implies that the marginal values for the units from the k-th to the q^* must be constant.

Proposition 3 also shows that if it is efficient to produce a positive number of units of the public good, then the largest Lindhal revenue of the firm is equal to the firm's VCG transfer when the two mentioned conditions hold, with the second one holding for all efficiently produced units of the public good.

Finally, Proposition 3 implies part (b) of Theorem 1 and provides the basis to generalize that result to an environment with multiple units and constant marginal values and marginal costs, as demonstrated by the next corollary.

Corollary 3. *Suppose that $v_i^1 = v_i^M$ for all $i \in \mathcal{N}$ and $c^1 = c^M$. Then, $\overline{R}_f = q^* \cdot \overline{\lambda}_f = \bar{t}_f$.*

Proof. By DMW and IMC, $v_i^1 = v_i^M$ for all $i \in \mathcal{N}$ and $c^1 = c^M$ imply that the marginal values of all agents and the marginal costs are constant. Let $v_i = v_i^1 = \ldots = v_i^M$ for each $i \in \mathcal{N}$ and let $c = c^1 = \ldots = c^M$. By definition, we have that $q^* = 0$ if $\sum_{j\in\mathcal{N}} v_j \leq c$ and $q^* = M$ if $\sum_{j\in\mathcal{N}} v_j > c$.

If $q^* = 0$, then $\overline{R}_f = q^* \cdot \overline{\lambda}_f = \underline{t}_f = 0$. If $q^* = M$, then $c^{q^*+1} = c^{M+1} = \infty$ and condition (a) in Proposition 3 holds. Since the marginal values of the agents are constant, condition (b) in Proposition 3 holds for $k = 1$. Then Proposition 3 implies that $\overline{R}_i = \overline{t}_f$. □

5 Producing a Public Bad

In this section, we sketch how our analysis applies to a pricing problem that can be viewed as the mirror image of the one studied so far: a firm derives some value from producing a public bad (e.g., air pollution) that induces a cost on some agents.[6] A classic example is the problem of a factory polluting a river while downstream residents need clean water for, e.g., agricultural production or domestic use (see e.g. Rob, 1989). In the digital age, the problem arises when a platform derives value from having access to the data of all its users,[7] but each user incurs a cost for giving up their privacy.[8]

Setup: Single-Unit Public Bad The firm derives value $v \in [0, \overline{v}]$ from producing a public bad, which imposes to each agent $i \in \mathcal{N}$ a cost $c_i \in [0, \overline{c}]$ with $\overline{v} \leq \overline{c}$.[9] As before, the agents have quasilinear utility in money; therefore, if the public bad is produced, the firm pays $\underline{t}_f$, and each agent i is paid $\overline{t}_i$, the payoffs are

$$v - \underline{t}_f \quad \text{and} \quad \overline{t}_i - c_i \quad \text{for each } i \in \mathcal{N}.$$

It is efficient for the public bad to be produced if

$$v > \sum_{i \in \mathcal{N}} c_i, \tag{11}$$

VCG transfers If (11) does not hold, then the public bad is not produced whether or not the agents and the firm participate; hence, the firm's and the agents' VCG transfers

[6] Mas-Colell et al. (1995, pp. 364-5) refer to such a public bad as a *nondepletable externatlity.*

[7] A user's data may only be valuable in conjunction with the data of all other users if the firm exploits patterns of correlation.

[8] The same problem arises in R&D when a firm—such as Apple for the iPhone—requires access to a multitude of patents or when a developer requires the consent of all land owners to build an apartment building or a mall.

[9] This assumption ensures that each agent has veto power and can prevent the production of the public bad by reporting a sufficiently high cost, thereby guaranteeing that each agent's individual rationality constraint is satisfied (her payoff is at least zero) under a VCG scheme. The firm's individual rationality constraint is satisfied because it can guarantee to make zero profit by reporting a value of zero.

are 0. Consider now the case where (11) holds. If the firm participates, each agent incurs a cost of c_i, while if the firm does not participate (by reporting the lowest value of 0), no cost is incurred. Therefore, the VCG transfer that the firm pays is $\underline{t}_f = \sum_{j \in \mathcal{N}} c_j$. If agent i participates, the public bad is produced, so the total payoff of other agents and the firm is $v - \sum_{j \neq i} c_j$. If agent i does not participate (by reporting the largest possible cost $\bar{c}$ and thus vetoing the production of the public bad), the public bad is not produced, so the total payoff of other agents and the firm is 0. Therefore, each agent i receives a VCG transfer of $\bar{t}_i = v - \sum_{j \neq i} c_j$.

Lindahl prices If (11) does not hold, then the public bad is not produced. Each agent i largest Lindahl revenue is 0 and the firm's smallest Lindahl expenditure is 0. Consider now the case where (11) holds. A vector $\boldsymbol{\lambda} \in \mathbb{R}^N_{\geq 0}$ is a Lindahl price vector if

- Each agent i is willing to incur the cost associated with the public bad: $\lambda_i \geq c_i$ for every $i \in \mathcal{N}$; and
- The firm is willing to produce the public bad: $\sum_{i \in \mathcal{N}} \lambda_i \leq v$.

The largest Lindahl price (and revenue) of each agent i is obtained by setting the price of every other agent $j \neq i$ to its lower bound c_j and satisfying the firm's constraint with equality; hence, $\overline{\lambda}_i = v - \sum_{j \neq i} c_j = \bar{t}_i$. The smallest Lindahl total price (and expenditure) of the firm is obtained by setting each agent's price to its lower bound; hence, $\underline{\lambda}_f = \sum_{j \in \mathcal{N}} c_j = \underline{t}_f$. The following mirror image of Theorem 1 follows.

Theorem 1*. *(a) The largest Lindhal revenue of each agent i is equal to agent i's VCG transfer, $\overline{R}_i = \underline{t}_i$ for every $i \in \mathcal{N}$, and if the public bad is produced, then the largest Lindhal price is also equal to the VCG transfer, $\overline{\lambda}_i = \overline{R}_i = \bar{t}_i$;*

(b) The smallest Lindahl expenditure of the firm is equal to the firm's VCG transfer, $\underline{E}_f = \bar{t}_f$, and if the public bad is produced, then the smallest Lindhal total price is also equal to the VCG transfer, $\overline{\lambda}_f = \underline{E}_f = \underline{t}_f$.

Mirror images of our other results can be obtained in a similar way. Therefore, the relationship between each participant's most favorable Lindahl price and the VCG transfer extends to environments with public bads.

6 Conclusions

This paper studies the relationship between VCG transfers and Lindahl prices in a setting with a single public good, which can be supplied by a firm in multiple units. The relationship is tightest if the public good must be provided as a single, indivisible unit. In that case, each agent's VCG transfer is equal to his smallest Lindahl price, and the firm's VCG transfer is equal to its largest Lindahl total price. We provide a symmetric result in an environment where the firm produces a public bad.

If the public good can be provided in multiple units, then the VCG transfer of an agent is equal to the sum of the marginal VCG transfers on each unit on which the agent is pivotal. In such a setting, the smallest Lindhal price of an agent is an upper bound on the agents marginal VCG transfer for each unit on which the agent is pivotal, and the largest Lindahl revenue of the firm is a lower bound on the firms VCG transfer.

An interesting avenue for future research would consist in studying the relationship between VCG transfers and Lindahl prices in more general settings. In particular, in a setting with multiple public goods, we conjecture that the lowest Lindahl expenditure of an agent is an upper bound on the agent's VCG transfer and the largest Lindahl revenue of a firm is a lower bound on the VCG transfer to the firm.

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